Probability Density Function

A random variable **X** is called continuous if its probabilities are described in terms of a nonnegative function f_X , called the **probability density function (PDF)** of X

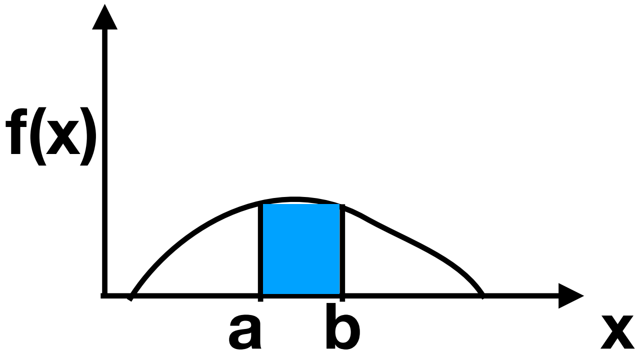
$$P(X \in B) = \int_{x \in B} f(x) dx$$

for every subset B of the real line.

$$\int_{-\infty}^{\infty} f_X(x) dx = 1 = (-\infty < P(X) < \infty)$$

The probability that **X** takes a value between **a** and **b** is

$$P(a \leq X \leq b) = \int_{a}^{b} f(x) dx$$



- Value of PDF at a particular point is zero: f(X=a)=0

- Interpretation of PDF: f(x)dx is the probability of finding the random variable X between x and x+dx.

- Example: A gambler's wheel. An example of *Continuous Uniform Random Variable*. Let the wheel is continuously calibrated between a and b, and has equal probability of obtaining any intermediate value.

$$f(x) = \begin{cases} c & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\int_a^b c dx = 1 \implies c = \frac{1}{b - a}$$

Expectation values

Expected value of g(X): $E[g(X)] = \int_a^b g(x) \underset{\uparrow}{f_X(x)} dx$

PDF of X

Central moments: $E[(X-\mu)^n] = \int_{x \in B} (x-\mu)^n f_X(x) dx$

CDF

- The cumulative distribution function of a continuous RV X with CDF f(x) is defined as :

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

Interpretation : Area under the curve of f(t) up to x

PDF from CDF: $\frac{dF}{dx} = f(x)$

Normal Random Variable

A continuous random variable X is said to be normal or Gaussian if it has a PDF of the form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

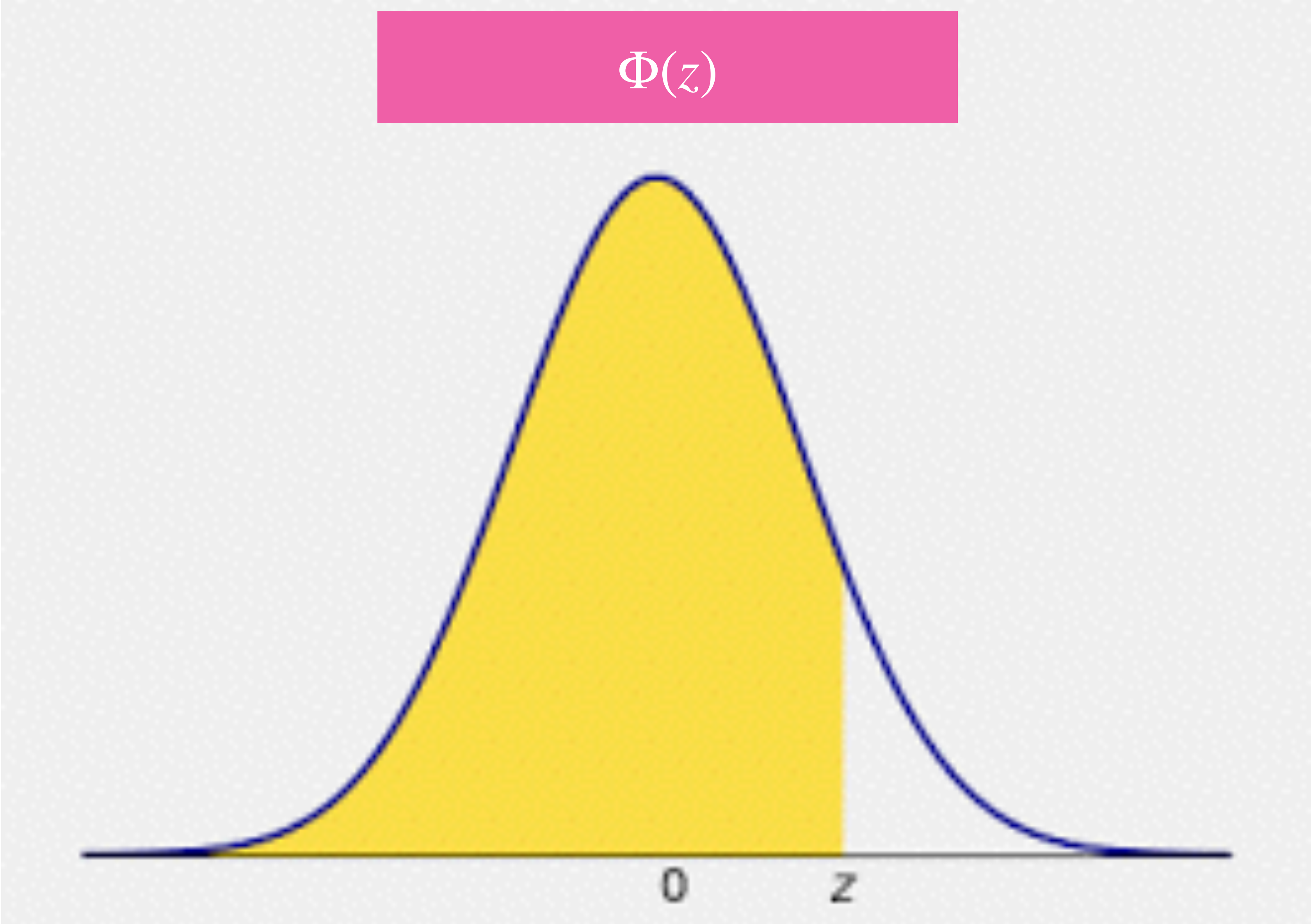
$$E[X] = \mu, V[X] = \sigma^2; \qquad X \sim N(\mu, \sigma^2)$$

CDF: $\Phi(x) = \int_{-\infty}^{x} f(y)dy$

Standard Normal PDF: A normal random variable X with zero mean and unit variance is said to be a standard normal. It is denoted by Z.

$$Z \sim N(0,1)$$

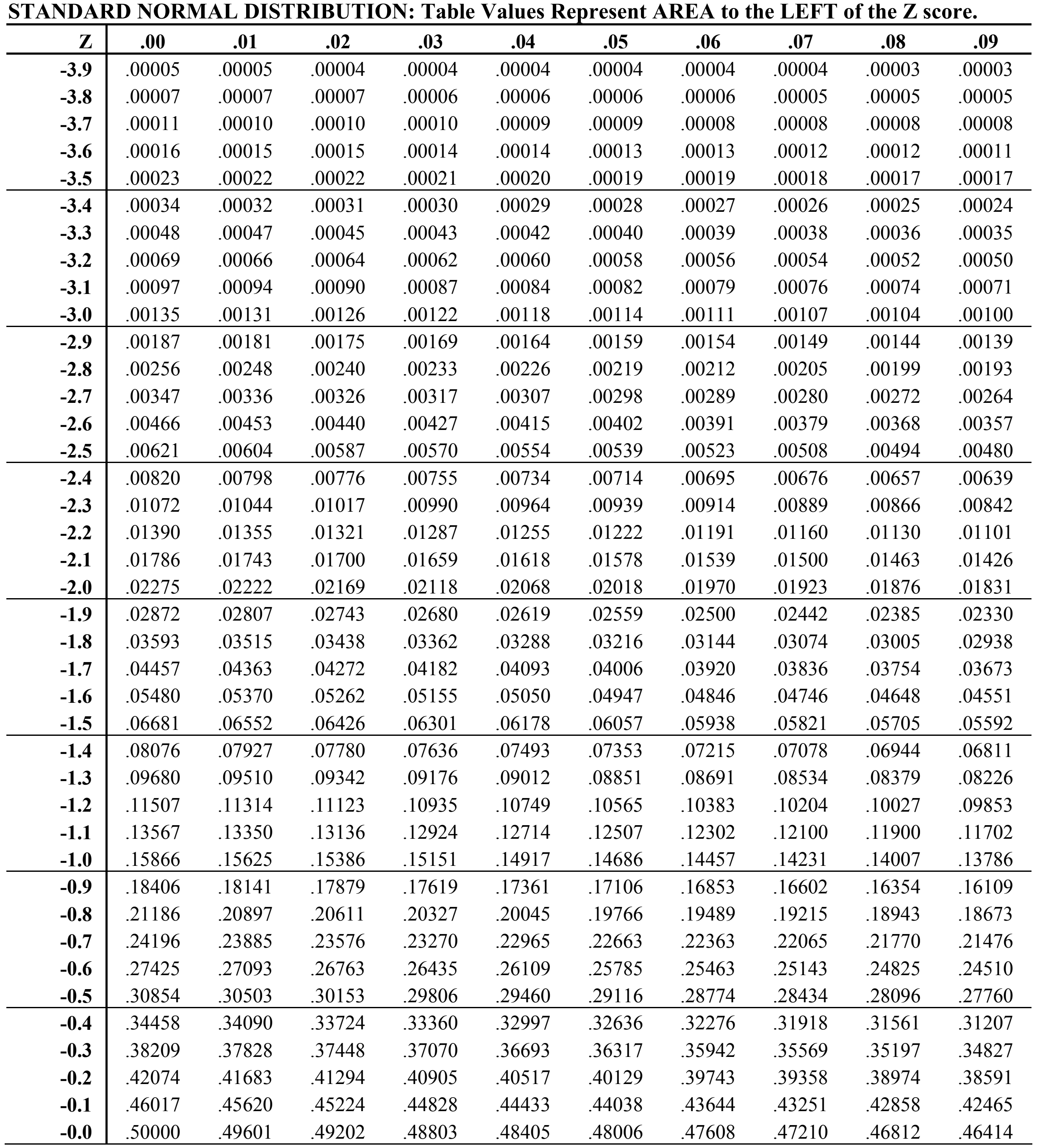
CDF: $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx$



CDF of Normal RV

Normal table

$$\Phi(z) + \Phi(-z) = 1$$



STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

CDF of standard normal density function

Example: Suppose a mild hypertensive is defined as a person whose DBP is between 90 and 100 mm Hg inclusive, and the subjects are 35- to 44-year-old men whose blood pressures are normally distributed with mean 80 and variance 144. What is the probability that a randomly selected person from this population will be a mild hypertensive?

- Ans. $\Phi(1.66) - \Phi(0.83) = .9522 - .7977 = .155$ (from table)