Calculus of Variations

We want to find a particular condition for a given expression (usually maximising or minimising) it by varying the functions on which the expression depends.

For instance, -

We want to know the ~~of~~ shape of a wire hanging from two fixed points for which it encloses ~~least~~ maximum area.

The problem is first expressed in mathematical form ~~y~~ → ~~I(x)~~ integral.

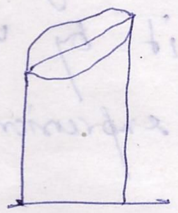
The problem may be formulated as below - we can find the shape by finding the shape for y(x) ~~the~~ by minimising the gravitational potential energy of the rope.

Each elementary piece of the rope has a gravitational pot. energy proportional both to its height from any arbitrary zero level and the length of the piece.

Total pot. energy is given by integrating all the elementary contributions of the pieces.

The particular function $y(x)$ for which the value of this integral is a minimum will give the shape assumed by the hanging rope.

The Euler-Lagrange equation



$$I = \int_a^b F(y, y'; x)\,dx \quad \text{—(1)}$$

a, b and the form of the function F are fixed by given considerations eg. physics of the problem. But the curve $y(x)$ has to be chosen so as to make stationary the value of I. which is clearly a functional.

we wish to find the function $y(x)$ such that first-order small changes in it will make only second-order changes in the value of I.

Let $y(x)$ is the function required to make I stationary and consider making the replacement.

$$y(x) \longrightarrow y(x) + \alpha\eta(x) \quad \text{—(2)}$$

where the parameter α is small and $\eta(x)$ is an arbitrary function with sufficiently amenable mathematical properties.

For the value of I to be stationary with respect these variations, we require

$$\frac{dI}{d\alpha}\Big|_{\alpha=0} = 0 \quad \text{for all } \eta(x) \longrightarrow (3)$$

$$I(y,\alpha) = \int_a^b F(y+\alpha\eta,\ y'+\alpha\eta',\ x)\,dx$$

$$= \int_a^b F(y, y', x)\,dx + \int_a^b \left(\frac{\partial F}{\partial y}\alpha\eta + \frac{\partial F}{\partial y'}\alpha\eta'\right)dx + O(\alpha^2)$$

For all $\eta(x)$ the ~~cond~~ to satisfy condition (3)

we require

$$\delta I = 0 + \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)dx = 0$$

where δI denotes the first-order variation in the value of I due to the variation in the function $y(x)$.

$$\frac{d}{d\alpha}(I) \Rightarrow \delta I$$

integrating the second term ~~ϕ~~ by parts this becomes

$$\left[\frac{\partial F}{\partial y'}\eta\right]_a^b - \int \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\,dx$$

$$+ \int_a^b \frac{\partial F}{\partial y}\eta(x)\,dx = 0$$

$$\Rightarrow \left[\frac{\partial F}{\partial y'}\eta\right]_a^b + \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\,dx = 0 \longrightarrow (4)$$

We assume end points are fixed

$\therefore \quad \eta(a) = \eta(b) = 0$

The first term of L.H.S. vanishes.

Since δI must be stationary so as (4) must be satisfied for arbitrary $\eta(x)$, it is easy to see that we require:

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{- Euler-Lagrange Equation}$$

<u>Special-cases</u>

(1) F does not contain y explicitly

So, $\dfrac{\partial F}{\partial y} = 0$

So, EL eqn becomes

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\text{or,} \quad \frac{\partial F}{\partial y'} = \text{constant}$$

F does not contain x explicitly

We multiply EL eqn by y'

using $$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + y''\frac{\partial F}{\partial y'}$$

$$y'\frac{\partial F}{\partial y} = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

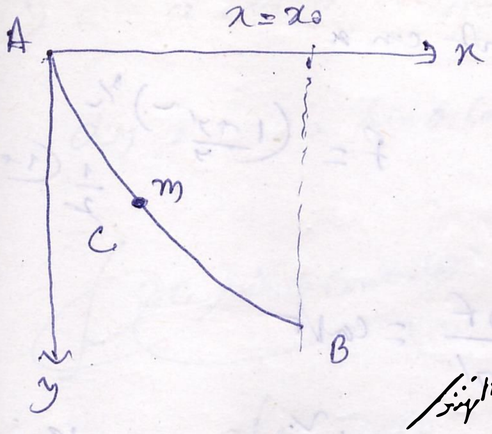
$$\Rightarrow y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)$$

L.H.S. $\Rightarrow$ $$\frac{dF}{dx} = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)$$

$$\boxed{F - y'\frac{\partial F}{\partial y'} = \text{const}}$$

Brachistochrone Problem:-

A frictionless wire in a vertical plane connects two points A and B, A being higher than B. Let the position of A be fixed at the origin of an xy-coordinate system, but allow B to lie anywhere on the vertical line $x = x_0$. Find the shape of the wire such that a ~~bead~~ bead of mass m placed on it at A will slide under gravity to B in the shortest possible time.



~~particle speed~~ ~~$v = \frac{ds}{dt}$~~ ~~at C along curve S after t.~~

$$v = \frac{ds}{dt}$$

$$ds = \left(1 + y'^2\right)^{1/2} dx$$

$$C \equiv (x, y) \qquad v^2 = 0 + 2gy$$

$$v = \frac{ds}{dt} = \sqrt{2gy} \qquad \therefore v = \sqrt{2gy}$$

The total time taken to travel to the line $x = x_0$ is given by

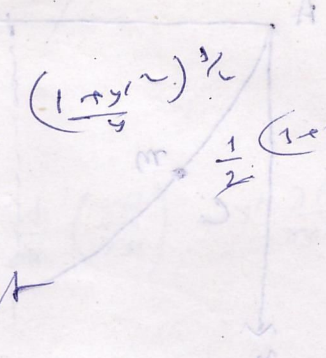
$$\int_{x=0}^{x=x_0} \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int_{x=0}^{x=x_0} \frac{\left(1+y'^2\right)^{1/2}}{y^{1/2}}\, dx$$

$$= \frac{1}{\sqrt{2g}} \int_0^{x_0} \sqrt{\frac{1+y'^2}{y}}\, dx$$

We have to extremise this integral.

EL equation $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

Since it doesn't con x



$y\frac{\partial f}{\partial y}$ $f = \left(\frac{1+y'^2}{y}\right)^{1/2}$ $\frac{1}{2}\left(1+y'^2\right)^{-1/2} \cdot 2y'$

$$f - y'\frac{\partial f}{\partial y'} = \text{const}$$

$$\sqrt{\frac{1+y'^2}{y}} - \frac{y'^2}{\left(y\left(1+y'^2\right)\right)^{1/2}} = K$$

$$\frac{1+y'^2 - y'^2}{\left[y\left(1+y'^2\right)\right]^{1/2}} = K$$

$$\left[y\left(1+y'^2\right)\right]^{1/2} = \frac{1}{K} = M$$

$$y' = \frac{dy}{dx}$$

$$\left[y\left(1+y'^2\right)\right] = M^2 = a$$

$$1+y'^2 = \frac{a}{y}$$

$$y'^2 = \frac{a}{y} - 1$$

$$y'^2 = \frac{a-y}{y}$$

$$y' = \sqrt{\frac{a-y}{y}}$$

$$\frac{dy}{dx} = \sqrt{\frac{a-y}{y}}$$

Canonical Equations of Hamilton

$$L = L(t, q_i, \dot{q}_i)$$

The system of these $t, q_i, \dot{q}_i$ specifies the state of a system at any instant.

Specification of the Lagrangian function and initial state uniquely determines the motion of the system.

Hamilton proposed another set of of variables t, q_i, p_i where p_i are generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (i = 1, \ldots, n).$$

t, q_i, p_i are called Hamiltonian variables.

Hamilton introduced the function $H(t, q_i, p_i)$ defined by

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L$$

$$\left.\begin{aligned} \frac{\partial H}{\partial q_i} &= -\dot{p}_i \\ \frac{\partial H}{\partial p_i} &= \dot{q}_i \end{aligned}\right\} \text{ Canonical equations or Hamilton's equation.}$$

① Derivation of Hamilton's Equations from Lagrange's Equations :-

$$H = H(p_k, q_k, t)$$

$$L = L(q_k, \dot{q}_k, t)$$

$$dH = \sum_k \frac{\partial H}{\partial p_k} dp_k + \sum_k \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial t} dt \quad -(1)$$

$$dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

$$dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k p_k d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

$$\therefore \quad p_k = \frac{\partial L}{\partial \dot{q}_k}$$

$$\text{and} \quad \dot{p}_k = \frac{\partial L}{\partial q_k} \qquad \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

$$\therefore \quad dL = \sum_k \dot{p}_k dq_k + \sum_k p_k d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

$$\cancel{dH =} \quad H = \sum_k p_k \dot{q}_k - L$$

$$dH = \sum_k dp_k \dot{q}_k + \sum_k p_k d\dot{q}_k - dL$$

$$= \sum_k dp_k \dot{q}_k + \sum_k \cancel{p_k} d\dot{q}_k - \sum \dot{p}_k dq_k - \sum \cancel{p_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt$$

$$= \sum_k dp_k \dot{q}_k - \sum \dot{p}_k dq_k - \frac{\partial L}{\partial t} dt \quad -(2)$$

Comparing eqn (1) and (2) weget

$$\left.\begin{aligned}\frac{\partial H}{\partial p_k} &= \dot{q}_k\\ \frac{\partial H}{\partial q_k} &= -\dot{p}_k\\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t}\end{aligned}\right\}$$

<u>Derivation of Hamilton's Equations from Hamilton's Principle</u> :-

$$\delta \int L\,dt = 0$$

$$H = \sum_k p_k \dot{q}_k - L \qquad H(t, p, q)$$

(1)

$$\delta \int L\,dt = \delta \int \left(\sum_k p_k \dot{q}_k - H \right) dt$$

$$= \cancel{\sum_k \int \left(\delta p_k \dot{q}_k + p_k \delta \dot{q}_k - \frac{\partial H}{\partial p_k}\delta p_k - \frac{\partial H}{\partial q_k}\delta q_k \right) dt}$$

$$= \sum_k \int \left[\dot{q}_k \delta p_k + p_k \delta \dot{q}_k - \frac{\partial H}{\partial p_k}\delta p_k - \frac{\partial H}{\partial q_k}\delta q_k \right] dt = 0$$

now $$\int_{t_0}^{t} p_k \delta \dot{q}_k\,dt = \underbrace{\left[p_k \delta q_k \right]_{t_0}^{t}}_{=0} - \int_{t_0}^{t} \dot{p}_k \delta q_k\,dt$$

$$\delta \int L\,dt = \sum_k \int^{t} \left[\left(\dot{q}_k \delta p_k - \frac{\partial H}{\partial p_k}\delta p_k \right) - \left(\frac{\partial H}{\partial q_k} + \dot{p}_k \right) \delta q_k \right] dt$$

$$= 0$$

each term ( ) must vanish identically

$$\frac{\partial H}{\partial p_K} = \dot{q}_K$$

$$-\left(\frac{\partial H}{\partial q_K} + \dot{p}_K\right) = 0$$

or, $$\frac{\partial H}{\partial q_K} = -\dot{p}_K$$