Classical Mechanics

IIIT, Allahabad

Instructor: Srijit Bhattacharjee

Principle of Least Action

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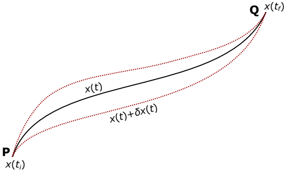
- PLA states, if one supplies the Lagrangian of a system along with the initial conditions (in particle mechanics, one needs to tell the gen. coordinates at initial (t_i) and final (t_f) times), then the trajectory that makes the Action stationary is given by Euler-Lagrange EOM or Newton's EOM.

How does the principle work?

Let us examine how the PLA works for a trajectory.

- Solving Newton's (or Lagrange's) equation of motion needs $2N$ number of initial conditions if the system has N dof. The N initial coordinates and N initial velocities.
- Given the initial condition(s) at an initial instant say at t_i, and the masses and Forces (or potential) of the system, one can solve the system to obtain the entire trajectory $x(t)$.



- For PLA, one provides the initial, and another point through which the system travels under the Forces present. For instant, if one tries to throw a stone through a tiny hole of a wall. The initial position and an intermediate position of the stone crossing the wall are fixed.

Extremizing the Action

- One writes the Action as:

$$\mathcal{A} = \int_{t_i}^{t_f} (K.E. - P.E.)dt.$$

  Remember that the PE and KE are both (implicit) functions of time. There can be imaginary paths that connect two terminal points. For each such different possible path one gets a different number for this action. Our goal is to find out for what curve that number is the least.

- Mathematically:

$$\delta\mathcal{A} = \delta \int_{t_i}^{t_f} \mathcal{L}(q_i, \dot{q}_i, t)dt = 0$$

- It is a remarkable fact that the path that *minimizes* the action is given by solving the Euler-Lagrange equation.

Calculus of Variation

- Action is not an ordinary function. It is actually a *functional*- a "function of a function"-here a function of all possible trajectories $(q_i(t))$.
- Extremizing a functional is a subject of a branch in mathematics called- the *calculus of variations.*
- Extremizing a function amounts to solve an equation (non-differential) $\implies$ stationary points. Extremizing a functional yields a differential equation $\implies$ trajectory.

Hamilton's principle

- Recall: $\delta \mathcal{A} = \delta \int_{t_i}^{t_f} \mathcal{L}(q_i, \dot{q}_i, t) dt = 0$
- We will make first order variations (change) in the $q_i s$ and $\dot{q}_i s$ to induce only second order change in the Action.
- The end point variations are zero $\implies$ $\delta q_i(t_i) = \delta q_i(t_f) = 0$.

$$L.H.S. \equiv \delta \int_{t_i}^{t_f} \mathcal{L}(q_i, \dot{q}_i, t) dt$$

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- Using: $\int_a^b \frac{d(f\delta x)}{dt} dt = f \delta x|_a^b = 0.$ for $\delta x(a) = \delta x(b) = 0$, and $\delta(\frac{d}{dt} q_i) = \frac{d}{dt}(\delta q_i).$

Euler-Lagrange Equation

- For arbitrary variations $\delta q_i s$, setting the last expression $= 0$, we get the E-L eqn.

$$\int_{t_i}^{t_f} \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} \right) \delta q_i dt = 0, \quad i = 1, 2, \cdots, N.$$

$\Longrightarrow$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

- Can accommodate certain non-holonomic systems and non-conservative forces also.

Variational Principle

- We find a particular condition for a given expression (usually maximising or minimising it) by varying the functions on which the expression depends.

$$I = \int_a^b F(y, y', x)dx \; ; \frac{\delta I}{\delta y} = 0$$

  Small change in $y(x)$ makes only second order change in I.

$$y(x) \to y(x) + \alpha\eta(x)$$

  End point variations are zero: $\eta(b) = \eta(a) = 0$.

- Taylor expand the function F up to first order. Throw away the boundary term and get the Euler-Lagrange equation:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

Two forms of Euler-Lagrange Equation

- EL equation when F **is independent of** y**:**

$$\frac{\partial F}{\partial y} = 0 \implies \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \to \frac{\partial F}{\partial y'} = const.$$

- EL equation when F **is independent of** x**:**
  Multiply EL eqn. by y'

$$y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - y'\frac{\partial F}{\partial y} = 0$$

with,

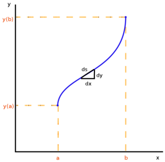
$$y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} = \frac{dF}{dx}.$$

this yields

$$y'\frac{\partial F}{\partial y'} - F = const.$$

Shortest Distance between two points in a plane

- element of length in a plane $ds = \sqrt{dx^2 + dy^2}$.



- The distance between two points (x_1, y_1) and (x_2, y_2) can be obtained by

$$I = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx = \int_{x_1}^{x_2} F(y') dx.$$

- EL eqn: $\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} = c$.
- On integration we get the eqn. straight line! $y = ax + b$.
- In general shortest curve between two points in any space is called a *Geodesic*.