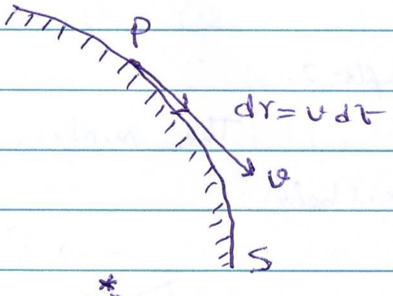
M T W T F S S

Date __/__/__

Virtual displacement

Example-1

A particle is in motion on a fixed surface.



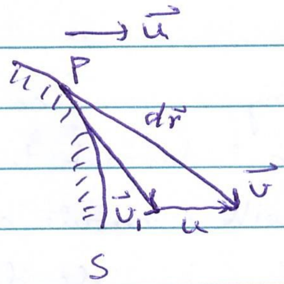
In this case any vector $\vec{v}$* constructed at pt. P on surface and tangent to the surface at that point will constitute a possible velocity. The corresponding possible displacement $d\vec{r} = \vec{v}\, dt$, lies in the plane tangent to the surface at pt. P. Take another possible displacement $d'\vec{r}$ and construct the virtual displacement $\delta\vec{r} = d'\vec{r} - d\vec{r}$. Incidentally $\delta\vec{r}$ also lies on the same tangent plane in which $d\vec{r}$ or $d'\vec{r}$ lies. Thus any vector lying in the tangent plane may be regarded as a certain $d\vec{r}$ and as a certain $\delta\vec{r}$.

* $\vec{v}$ is one of the possible velocities.

Here the constraint is stationary and the virtual displacements coincide with possible displacements.

Example 2

The surface is itself in motion (as a rigid body).



Let us now consider the surface S is moving with a certain velocity $\vec{u}$ relative to the original system of coordinates. In this case possible velocity $\vec{v}$ is obtained from an arbitrary vector $\vec{v}_1$ that is tangent to the surface by adding the ~~relative~~ velocity $\vec{u}$ to it. $\vec{v} = \vec{v}_1 + \vec{u}$

Therefore, $d\vec{r} = \vec{v}\,dt = \vec{v}_1\,dt + \vec{u}\,dt$

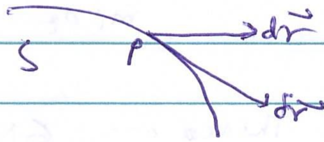
Similarly, another possible displacement

$$d'\vec{r} = \vec{v}_1' \, dt + \vec{u} \, dt$$

So, the virtual displacement

$$\delta\vec{r} = d'\vec{r} - d\vec{r} = (\vec{v}_1' - \vec{v}_1) \, dt$$

Although $\delta\vec{r}$ is lying in a plane tangent to the surface at the pt P, but it is not coplanar with $d\vec{r}$. $d\vec{r}$ doesn't lie in the plane tangent to P.



Recall virtual displacements in general satisfy following set of constraints

$$\sum_{i=1}^{N} \frac{\partial f_k}{\partial \vec{r}_i} \delta\vec{r}_i = 0 \qquad (vi) \quad (k = 1, 2, \dots, m)$$

and

$$\sum_{i=1}^{N} \vec{l}_{pi} \, \delta\vec{r}_i = 0 \qquad (vii) \quad (p = 1, 2, \dots n)$$

The equation of motion for i th particle in the system may be expressed as

$$m_i \vec{a}_i = \vec{F}_i^{(a)} + \vec{f}_i \quad \dots (viii)$$

Where $\vec{F}_i^{(a)}$ is called effective force or applied force and $\vec{f}_i$ is termed as reaction force or force of constraints. and $\vec{a}_i$ is the ~~possible~~ acceleration.

In cartesian co-ordinate system one may write

$$\left.\begin{aligned} m_i a_x &= F_{ix}^{(a)} + f_{ix} \\ m_i a_y &= F_{iy}^{(a)} + f_{iy}, \\ m_i a_z &= F_{iz}^{(a)} + f_{iz} \end{aligned}\right\} \quad (ix)$$

There are 6N unknowns (3N, x, y, z co-ordinate and 3N forces of constraints)

but no of equations available is $3N + m + n$.

No of unknowns > no of eqns $[6N > 3N + m + n]$

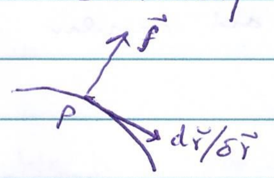
We need ~~[illegible]~~ $r = 3N - m - n$ additional relations to be able to completely determine the system. This is achieved by confining ourselves to constraints that obey the condition that net virtual work of the forces of constraints is zero.

$$\therefore \sum_{i=1}^{N} \vec{f}_i \cdot \delta \vec{r}_i = 0 \quad \ldots \quad (x)$$

This kind of situation is not unrealistic as can be seen in following examples.

Ex:-1

A particle is constrained to move on a stationary smooth surface.



In this case force of constraint at any pt P is directed normally to the possible/virtual displacements.

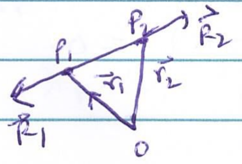
$$\therefore \quad \vec{f}\cdot\delta\vec{r} = 0 = \vec{f}\cdot d\vec{r}$$

Ex-2 Particle is constrained to move on a mobile or deforming smooth surface.

$$\vec{f}\cdot\delta\vec{r} = 0 \text{ but } \vec{f}\cdot d\vec{r} \neq 0.$$

More non-trivial example

Two particles are connected by a rod of invariable length and of negligible mass.



$\vec{R}_1$ and $\vec{R}_2$ are forces of contraints impressed on particles P_1 and P_2.

M T W T F S S
Date __ / __ / __

By Newton's third law, the rod is acted upon by forces $\vec{F_1} = -\vec{R_1}$, $\vec{F_2} = -\vec{R_2}$.

Let m, $\vec{a}$ be the mass and acceleration of the rod's centre of mass. and I and $\vec{\alpha}$ be the moment of inertia and angular acceleration. ~~of the~~ Then

$$m\vec{a} = \vec{F_1} + \vec{F_2}$$

and
$$\underset{\text{Torque.}}{\vec{L}} = I\vec{\alpha}$$

It is given $m = 0$ and $I = 0$.

$\therefore$ $\vec{F_1} = -\vec{F_2}$. and $\vec{L} = 0$.

Further, $\vec{R_1}\cdot\delta\vec{r_1} + \vec{R_2}\cdot\delta\vec{r_2} = \vec{R_1}\cdot d\vec{r_1} + \vec{R_2}\cdot d\vec{r_2}$

$\Rightarrow \vec{R_1}(d\vec{r_1} - d\vec{r_2}) = \vec{R_1}\cdot d(\vec{r_1} - \vec{r_2})$ $\quad \because \vec{R_1} = -\vec{R_2}$

Let $\vec{R_1} = c(\vec{r_1} - \vec{r_2})$

then $\vec{R_1}\cdot d(\vec{r_1} - \vec{r_2}) = c(\vec{r_1} - \vec{r_2})\cdot d(\vec{r_1} - \vec{r_2})$

$$= \frac{c}{2} d(\vec{r_1} - \vec{r_2})^2 = 0$$

Since $(\vec{r_1} - \vec{r_2})^2 = \text{const}$

$\therefore$ $\boxed{\vec{R_1}\cdot\delta\vec{r_1} + \vec{R_2}\cdot\delta\vec{r_2} = 0}$

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When constraints of these sort are present in the system i.e. constraints obeying $\vec{f_i} \cdot \delta\vec{r_i} = 0$, we may write general eqn. of dynamics as

$$\sum_{i=1}^{N} \left(\vec{F}_i^{(a)} - m_i \vec{a_i}\right) \cdot \delta\vec{r_i} = 0 \quad (xi)$$

We have got rid off the forces of constraints but still can't equate the co-efficients of all $\delta\vec{r_i}$'s to zero as not all $\delta\vec{r_i}$'s are independent. (Still they are not written in terms of generalized co-ordinates)

Since,

$$\sum_{i=1}^{N} \vec{f_i} \cdot \delta\vec{r_i} = 0$$

one may write this as

$$\sum_{i=1}^{N} \left(\vec{f_i} - \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial \vec{r_i}} - \sum_{p=1}^{n} \beta_p \vec{\phi}_{pi}\right) \cdot \delta\vec{r_i} = 0 \quad (xii)$$

λ_k and β_p's are Lagranges multiplier

One can choose these $m+n$ number of multipliers λ_k and β_p so that in eqn (xii) the co-efficients of $m+n$ independent increments vanish. Then one can express (xii) as

$$\vec{F_i} = \sum_k \lambda_k \frac{\partial f_k}{\partial \vec{r_i}} + \sum_p \beta_p \, l_{pi}$$

now, (viii) $\Rightarrow$

$$m_i \vec{a_i} = \vec{F_i}^{(a)} + \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial \vec{r_i}} + \sum_p \beta_p \vec{l_{pi}}$$

$\hookrightarrow$ Lagrange's eqn of first kind.

along this one must add these equation.

$$f_k(\vec{r_i}) = 0, \quad \sum_{i=1}^{N} l_{pi} \vec{v_i} + D_p = 0$$
$(k=1,2,\cdots,m)$ $\qquad$ $(p=1,2,\cdots,n)$

In Total: system contains $3N+m+n$ unknowns.

and $3N+m+n$ equations

For such large number of equations, ~~that~~ Lagrange's equation of first kind of little use.

Lagrange's equation of motion of second kind:-

Principle of virtual work:-

If the system is at equilibrium, then total force on it's ith particle must be zero and total virtual work must be zero.

$$\vec{F}_i \cdot \delta\vec{r}_i = 0 \qquad (xiii)$$

If we confine ourselves to the ~~f~~ systems when constraint forces do not perform any virtual work

then (xiii) $\Rightarrow$ $\vec{F}^{(a)} \cdot \delta\vec{r}_i = 0$. (xiv)

Eq. (xiv) is known as principle of virtual work.

This statement (eq (xiv)) is not suitable for dynamics as it is a ~~system~~ statement of a system at equilibrium (static). One may see the general equation of dynamics (xi) in a different way to generalise the eqn. (XIV).

~~Let us~~.

The general equation of dynamics may be considered as an equation expressing the principle of virtual work and characterizing the static state of the system under the influence of ~~total~~ ~~applied~~ ~~effective~~ * applied forces $\vec{F}_i^{(a)}$ and a fictitious 'reverse effective forces'. $-\dot{\vec{p}}_i = -m_i \vec{a}_i$.

i.e. $\left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0$ (XV)

This is called as D' Alembert's principle.

This principle permits extending the techniques and methods of solution of static problems to problem of dynamics.

We need to transform eq. (XV) in terms of generalized co-ordinates.

M T W T F S S

Date __/__/__

The relation between two set of co-ordinates:

$$\vec{r_i} = \vec{r_i}(q_1, q_2, \ldots, q_r, t) \quad \ldots (xvi)$$

r-independent co-ordinates.

$\vec{v_i} \equiv$ velocities can be expressed as

$$\vec{v_i} = \frac{d\vec{r_i}}{dt} = \sum_{k=1}^{r} \frac{\partial \vec{r_i}}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t} \quad \ldots (xvi)$$

Connection between virtual displacements of two co-ordinates systems :

$$\delta \vec{r_i} = \sum_{j=1}^{r} \frac{\partial r_i}{\partial q_j}\delta q_j \quad \ldots (xvii)$$

Virtual work $\Rightarrow \sum_i \vec{F_i}^{(a)} \cdot \delta\vec{r_i} = \sum_{i,k} \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_k}\delta q_k$

[one can drop the superscript 'a' on force as we have employed D'Alembert's principle].

$$= \sum_k Q_k \delta q_k$$

Q_k are called components of generalized force.

$$Q_k = \sum_i \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_k} \quad \ldots (xviii)$$

Next,

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i$$

$$\Rightarrow \sum_{i,k} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}\,\delta q_k$$

now, $$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}\right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right)\right]$$

— (xix)

now, $$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_k}$$

$$= \sum_{j=1}^{r} \frac{\partial^2}{\partial q_k \partial q_j} + \frac{\partial^2 \vec{r}_i}{\partial q_k \partial t}$$

$$= \frac{\partial \vec{v}_i}{\partial q_k}$$

further $$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$ (Since from (xvi) $$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\delta_{kj}$$)

M T W T F S S

Date __ / __ / __

Substituting these in (XIX)

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_k}\right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j}\right] \quad \cdots (XX)$$

now, $m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} = \frac{1}{2} \frac{\partial}{\partial q_j}\left(m_i \vec{v}_i^{\,2}\right)$

So, eqn (XIX) (R.H.S) $\Rightarrow \sum_i \frac{d}{dt}\left[\frac{\partial}{\partial q_k}\left(\frac{1}{2} m_i v_i^2\right)\right] - \frac{\partial}{\partial q_k}\left(\frac{1}{2} m_i v_i^2\right)$

Now, eqn (XV) $\Rightarrow$

$$\cancel{\sum_k Q_k - \left(\frac{d}{dt}\frac{\partial}{\partial q_k}\frac{1}{2} m_i v_i^2\right)}$$

$$\sum_k Q_k - \left\{\frac{d}{dt}\left[\frac{\partial}{\partial q_k}\left\{\sum_{i=1}^{N}\left(\frac{1}{2} m_i v_i^2\right)\right\}\right] - \frac{\partial}{\partial q_k}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right\} \cdot \delta q_k = 0$$

next, identify $T = \sum_i \frac{1}{2} m_i v_i^2 \rightarrow$ kinetic energy of the system

then D'Alembert's principle implies

$$\sum_k \left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_k}\right) - \frac{\partial T}{\partial q_k} - Q_k\right] \cdot \delta q_k = 0 \quad (XXI)$$

~~equations~~.

Further if we restrict to holonomic system of constraints, then it is possible to give an elementary virtual displacement to k^{th} co-ordinate q_k while keeping remaining independent co-ordinates unchanged.

Therefore only way for (xxi) to hold is the individual co-efficients vanish.

$$\therefore \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \quad \text{--(xxii)}$$

↳ L.E of 2nd kind.

total r- such equations.

If forces are derivable from a scaler potential V,

$$\vec{F_i} = -\vec{\nabla}_i V$$

then,

$$Q_j = \sum_i \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_j}$$

$$= -\sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r_i}}{\partial q_j}$$

$$= -\frac{\partial V}{\partial q_j}$$

M T W T F S S

Date __/__/__

Then, (xxii) $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 \quad \cdots (xxiii)$$

If V is independent of time then system is conservative. [V is anyway independent of generalized velocities]

Then one can write (xxiii) as

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

define Lagrangian $L = T-V$,

we get,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \quad (xxiv)$$

$\hookrightarrow$ Lagrange's equations

M T W T F S S
Date __/__/__

The choice of Lagrangian is not unique. Many possible Lagrangians result in same equations of motion. Addition of any total derivative of a function of generalized co-ordinates and time, to a Lagrangian yields same equation of motion.

If $$L_{new} = L_{old} + \frac{dF(q,t)}{dt}$$

then as $$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t}$$

and $$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\left(\frac{dF}{dt}\right) = \frac{d}{dt}\left(\frac{\partial F}{\partial q}\right)$$

one has $$\frac{d}{dt}\left(\frac{\partial L_{new}}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial L_{old}}{\partial \dot{q}}\right) + \frac{d}{dt}\left(\frac{\partial F}{\partial q}\right)$$

and $$\frac{\partial L_{new}}{\partial q} = \frac{\partial L_{old}}{\partial q} + \cancel{\frac{d}{dt}\left(\frac{\partial F}{\partial q}\right)}\,\frac{\partial}{\partial q}\left(\frac{dF}{dt}\right)$$

Therefore $$\frac{d}{dt}\left(\frac{\partial L_{new}}{\partial \dot{q}}\right) - \frac{\partial L_{new}}{\partial q} = \frac{d}{dt}\left(\frac{\partial L_{old}}{\partial \dot{q}}\right) - \frac{\partial L_{old}}{\partial q}$$

So, the Lagrange's equation of motion remains unchanged!

Let us consider the case when the potential depends upon generalized velocities and time also. This is now called generalized potential $\tilde{V}(q_k, \dot{q}_k, t)$.

The generalized force now be expressed as

$$Q_k = \sum_i \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_k}$$

$$= \frac{d}{dt}\left(\frac{\partial \tilde{V}}{\partial \dot{q}_k}\right) - \frac{\partial \tilde{V}}{\partial q_k}, \quad k = 1, \dots, n$$

now, if we define $L = T - \tilde{V}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial \tilde{V}}{\partial \dot{q}_k}\right) - \frac{\partial \tilde{V}}{\partial q_k}$$

or, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad !!$$

M T W T F S S

Date __/__/__

Electromagnetic force on moving charges :-

This is an example of velocity dependent potential.

a particle with mass m and charge q is moving at a velocity $\vec{v}$.

The force acting on it will be given by

$$\vec{F} = q\left[\vec{E} + (\vec{v} \times \vec{B})\right]$$

$\vec{E}(t, x, y, z)$ and $\vec{B}(t, x, y, z)$ are Electric and magnetic fields and can be derivable from a scalar potential $\phi(t, x, y, z)$ and a vector potential $\vec{A}(t, x, y, z)$ respectively by

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

and $$\vec{B} = \vec{\nabla} \times \vec{A}$$

M T W T F S S

Date __/__/__

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

The force on the charge can be derivable from a potential

$$\tilde{V} = q\phi - q\vec{A}\cdot\vec{v}$$

and, $L = T - \tilde{V}$

Lagrangian $L = \frac{1}{2}mv^2 - q\phi + q\vec{A}\cdot\vec{v}$

Eqn of motion

$$\frac{\partial L}{\partial v_x} = m v_x + q A_x$$

$$\frac{\partial L}{\partial x} = -q\frac{\partial \phi}{\partial x} + q\frac{\partial A_x}{\partial x}v_x + q\frac{\partial A_y}{\partial x}v_y + q\frac{\partial A_z}{\partial x}v_z$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right) - \frac{\partial L}{\partial x} = m\dot{v}_x + q\dot{A}_x + q\frac{\partial \phi}{\partial x} - q\frac{\partial A_x}{\partial x}v_x - q\frac{\partial A_y}{\partial x}v_y - q\frac{\partial A_z}{\partial x}v_z = 0 \quad \rightarrow (1)$$

$$\frac{dA_x}{dt} = \dot{A}_x = \frac{\partial A_x}{\partial t} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}$$

now, (1) $\Rightarrow$
$$m\dot{v}_x = -q\frac{\partial \phi}{\partial x} - q\frac{\partial A_x}{\partial t} + q\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)v_y + q\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)v_z$$

$$= -q(\vec{\nabla}\phi)_x - q\left(\frac{\partial \vec{A}}{\partial t}\right)_x + q[\vec{v}\times(\vec{\nabla}\times\vec{A})]_x$$

Hence $F_x = q[E_x + (\vec{v}\times\vec{B})_x]$