Review of Newtonian Mechanics

Classical mechanics deals with motion of 'particles' or 'system of particles'.

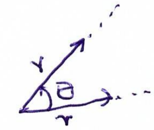
The term 'particle' (a point) is a fundamental concept of mechanics. It means a body whose dimensions may be ignored & in describing it's motion.

The possibility of doing so of course depends on the particular problem at hand. For example, in planetary motion around a star, planets may be considered as particles. But if we are interested to the rotational motion of planets about their axes, we can not approximate them as particles.

To consider any mechanical phenomena we need to choose a frame of reference.

In general the laws of motion are different in different frames of reference. Sometimes, the laws governing the motions of a simple system become complex if we don't choose an appropriate referance frame.

Some times space may not remain homogenous and isotropic. A body which is isolated from all interactions, ~~may should~~ could



not remain at rest. If at some instant it's velocities were zero, it can start moving in some direction at the next instant.

Same thing can happen for time also. Time may not remain ~~isotropic~~ homogeneous in arbitrary frame. This would mean, different instances will not be equivalent!

We must avoid this kind of situation.

However, it is always possible to find a frame of reference in which space is homogeneous and isotropic and time is homogeneous. This is called as '<u>inertial frame</u>'.

A body which is not subjected to any external action, if is found to be at rest or in uniform in one instant, remains at rest always.

This is Newton's first law!

Existence of inertial reference frame.

~~First law~~ Second Law :- Introduces concepts of mass, acceleration and force (quantitative definition).

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In an inertial frame an body moving with constant velocity continues to move without changing the velocity if it is not acted on by a force.

This is Newton's first law!

Inertial reference frame exists.

First law tells us qualitative nature of 'force'. ~~As~~ It is the thing which is responsible for not allowing an object to move uniformly in any reference system which is inertial.

Second Law :-

Second law introduces concepts of mass, acceleration and force — quantitative definition. It is written as

$$m \frac{d^2\vec{r}}{dt^2} = \vec{F} \text{ (force)}$$

$$\frac{d^2\vec{r}}{dt^2} = \vec{a} = \text{acceleration of the particle}$$

m = mass of the particle.

It is important to realize that force is not merely a matter of definition. If an isolated system suddenly starts to move with an acceleration $\vec{a}$, one may tempt to conclude that a force is acting on it $\vec{F} = m\vec{a}$. But this may not be true as forces arise only due to real physical interactions between bodies. Interactions are significant and accelerations are merely consequence of interactions. For example, if one moves in an accelerating system (e.g. circular motion) (s)he is in an non-inertial frame and may measure a non-zero acceleration of an object which is moving uniformly with respect to an inertial frame. But (s)he should not conclude that the object is acted on by a force, as Newton's law is not applicable in non-inertial frames. We usually call this kind of force as 'fictitious' force as this has not arised because of any physical interactions. If impossible we can eliminate all the interactions from the body — then it will indeed be possible to conclude that the body is moving uniformly in an inertial system.

One may wonder if complete isolation is possible in real world! But we are fortunate, that most common interactions that we encounter, like electromagnetic or gravitational interactions die at large distances. For example, Coulomb force decreases as $1/r^2$. So as gravitational. Forces between separated atoms decrease as $1/r^7$. So, for all practical purposes one may quite convincingly achieve a complete isolation.

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Force must be a result of interaction is emphasized in Newton's third law. All forces should appear in pairs if they are result of physical interactions. If body a exerts a force $\vec{F}_b$ on body b then body b should also exert a force that is equal and opposite to $\vec{F}_b$. In other words,

$$\vec{F}_a = -\vec{F}_b .$$

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Conservation Laws:- Ref: (Goldstein's book, class notes).

Single particle :-

If total force on a particle is zero, then the linear momentum of the particle is conserved. (Conservation of linear momentum)

$$\vec{F} = \frac{d\vec{p}}{dt} = 0$$

$$\Rightarrow \quad \vec{p} = \text{const.}$$

If total torque on the particle is zero, the angular momentum is conserved.

$$\vec{N} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = 0$$

$$\Rightarrow \quad \vec{L} = \text{const.}$$

Work-Energy relation:-

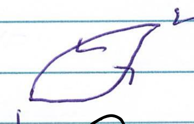
$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} = \frac{m}{2}\left(v_2^2 - v_1^2\right) = T_2 - T_1$$

Work done by the particle = change in kinetic energy.

Conservative force :-

The work done by the particle ~~going~~ moving from pt 1 and 2. is independent of path if the force is conservative. i.e. no dissipation is taking place. In this case:

* $$\oint \vec{F}.d\vec{s} = 0$$



and $$\vec{F} = -\vec{\nabla} V(r)$$

$V(\bar{r})$ is called potential

* $\vec{\nabla}$ is gradient operator.

In cartesian system :

$$\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

For conservative systems :

$$W_{12} = V_1 - V_2$$

Energy conservation theorem:-

If forces acting on particle are conservative, then the total energy of the particle, T + V, is conserved

$$W_{12} = T_2 - T_1 = V_1 - V_2$$

$$\Rightarrow \quad T_1 + V_1 = T_2 + V_2 .$$

* gradient of any function (scalar) points in the direction of maximum increase of the function.

* Stoke's theorem :-

$$\cancel{\int \vec{\nabla} \times \vec{F} \cdot d\vec{a}}$$

$$\oint_{\text{Surface}(A)} \vec{\nabla} \times \vec{F} \cdot d\vec{a} = \oint_{\text{line}} \vec{F} \cdot d\vec{l}$$

$\vec{\nabla} \times \vec{F}$ is called curl of vector $\vec{F}$.

the line integral is performed along the boundary of the surface A.

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Mechanics of system of particles:

A system of N particles is acted upon external and internal forces. Force on the ith particle can be written as

$$\vec{F} = \underbrace{\vec{F}_i^{(e)}}_{\text{external force}} + \underbrace{\sum_j \vec{F}_{ji}}_{\text{internal force on ith particle due to the jth particle.}} = \dot{\vec{P}}_i$$

We shall assume internal forces (F_{ij}s) obey Newton's third law of motion. Sometimes this is referred to as the weak law of action and reaction.

Summed over all particles,

$$\sum_i \vec{F}_i = \sum_i \vec{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \vec{F}_{ji} = \sum_i \vec{F}_i^{(e)}$$

since $F_{ij} + F_{ji} = 0$

Centre of Mass :- $\vec{R} = \frac{\sum_i m_i \vec{r_i}}{\sum m_i} = \frac{\sum_i m_i \vec{r_i}}{M}$
(C.O.M.)

Eqn. of motion $M\ddot{\vec{R}} = \sum_i F_i^{(e)}$

Conservation law for linear momentum:

If total external force is zero, total linear momentum $\vec{P} = M\dot{\vec{R}}$ is conserved.

Total torque :-

$$\vec{N} = \sum_i \vec{r_i} \times \vec{F_i} = \dot{\vec{L}} = \sum_i \vec{r_i} \times \vec{F_i}^{(e)}$$

$$+ \sum_{\substack{i,j \\ i \neq j}} \vec{r_i} \times \vec{F_{ji}}$$

Int. Forces act along line of their joining $\rightarrow$ Strong law of action and reaction.

So, second term in total torque vanishes.

$$\frac{d\vec{L}}{dt} = \vec{N}^{(e)}$$

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$\vec{L}$ is conserved if total external torque is zero.

The dynamics of system of particles can be described as sum of ~~two~~ contributions of motion of C.O.M. and motion about the C.O.M.

Eg:- $\vec{L} = \vec{R} \times M\vec{V} + \sum_i \vec{r_i}' \times \vec{p_i}'$

Total kinetic energy:-

$$T = \frac{1}{2}M\vec{V}^2 + \frac{1}{2}\sum_i m_i v_i'^2$$

If both the external and internal forces are derivable from ~~a~~ potentials then total potential energy can be expressed as.

$$V = \sum_i V_i + \frac{1}{2}\sum_{\substack{i,j \\ i \neq j}} V_{ij}$$

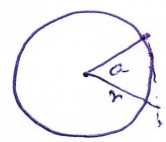
In absence of non-conservative forces the total energy for system of particles is conserved and equal to $T+V$.

Constraints

Eqns. of motion for a system particles may be represented as the set of differential equations

$$m_i \ddot{\vec{r}} = \vec{F}_i^{(e)} + \sum_j \vec{F}_{ji} \quad \cdots \quad (A)$$

However, practically this set of equations can not describe the actual motion of particles. There are constraints on the system, that allow only a subset of all possible motions. For example a particle placed on the surface of a solid sphere can only move along the surface or outside the surface of the sphere.



Constraint :- $r^2 - a^2 \geq 0$

Constraints (definition):- Let us consider a system of N particles whose position vectors are $\vec{r}_1, \vec{r}_2, \cdots \vec{r}_N$ with respect to some inertial system ~~of coordinates~~ or reference frame. The restrictions (geometric or kinematical) imposed on the positions and velocities of the particles are called constraints.

Analytically a constraint is expressed by the equation of type :

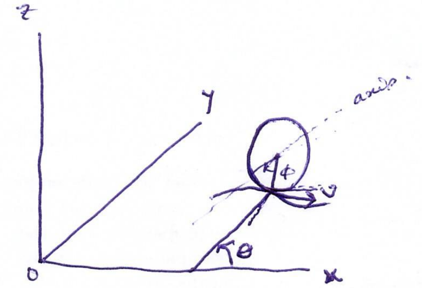
$$f(t, \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, \dot{\vec{r}}_1, \dot{\vec{r}}_2, \ldots, \dot{\vec{r}}_N) = 0 \quad \cdots (1)$$

Eqn. (1) is called <u>differential</u> or <u>kinematical</u> constraint. Here velocities enter into the constraint equation. When this is not the case, the constraint is called <u>geometric</u> or <u>finite</u>. Analytically it is written as

$$f(t, \vec{r}_1, \ldots, \vec{r}_N) = 0 \quad \cdots (2)$$

Example of differential constraint :-
(Kinematic)

Vertical disk rolling on a plane.



angle of rotation = ϕ (about the axis)
radius of the disk = a
angle between x axis and axis of the disk = θ
(x_0, y_0) is the co-ordinate of the centre of the disk.

Since the disk is moving on the x-y plane and always remains vertical w.r.t. the plane, the velocity of the centre of the disk, is $\vec{v}$, and it's magnitude is given by,

$$v = a\dot{\phi} \quad \ldots (a)$$

and,
$$v_x = \dot{x} = v \sin\theta$$
$$v_y = \dot{y} = -v \cos\theta \quad \text{--- (b)}$$

combining two sets of eqns.

$$dx - a\sin\theta \, d\phi = 0 \quad \ldots (c)$$
$$dy + a\cos\theta \, d\phi = 0$$

The constraints (c) are called differential. However, they are not integrable. Therefore it cannot be reduced to the form eq. (2). So this is not a geometric or finite constraint.

Example of geometrical constraint :- A particle is constraint to move over a surface and the surface is moving or undergoing deformation.

$$f(t, \vec{r}) = 0 \quad \rightarrow \text{finite / geometrical constraint}$$
$\downarrow$
eqn. of surface.
$$\equiv f(t, x, y, z) = 0$$

We are interested only on those constraints (differential) whose equation ~~of motion~~ contain the velocities of the particles in linear form :-

$$\sum_{i=1}^{N} \vec{l}_i \cdot \dot{\vec{r}}_i + D = 0$$

The vector $\vec{l}_i$ and D are specified functions of t and of all $\vec{r}_i$'s.

Finite constraints impose restrictions to possible positions of the system at time t.

With differential constraint alone, a system may occupy any arbitrary position in space at any time t. but the velocities are no longer arbitrary.

If the geometrical or finite constraint does not include time 't' explicitly in the constraint equation, then it is called <u>Sceleronomous</u> constraint.

Eg:- Rigid body motion,

$$(\vec{r}_i - \vec{r}_j)^2 - c_{ij}^2 = 0$$

From, (2) one can write

$$\frac{df}{dt} = 0 \Rightarrow \sum_{i=1}^{N} \frac{\partial f}{\partial \vec{r}_i} \cdot \dot{\vec{r}}_i + \frac{\partial f}{\partial t} = 0 \quad (3)$$

For <u>Sceleronomous</u> constraint $\frac{\partial f}{\partial t} = 0$

$\therefore$ (3) becomes

$$\sum_{i=1}^{N} \frac{\partial f}{\partial \vec{r}_i} \cdot \dot{\vec{r}}_i = 0 \quad (4)$$

This equation is linear and homogeneous in velocities.

A constraint is rheonomic if it includes time dependence in constraint equation. Eg - 2nd example

4)

A differential constraint is stationary or scleronomous if $D = 0$ and the vector $\vec{l_i}$, $\vec{r_i}$ are independent of time.

A ~~system~~ constraint is called holonomic if the particles of the system are not subjected to differential non-integrable constraints.

Eg :- (a) two particles connected by rod of length l

$$(\vec{r_1} - \vec{r_2})^2 - l^2 = 0$$

(b) particle moving on a surface and surface is deforming :-

$$f(t, x, y, z) = 0$$

Non-holonomic constraints are those which can't be integrable :- Eg :- (a) first example (one has to solve the system completely to cast it as eq (2))

(b) particle placed on the surface of a sphere $r^2 - a^2 \geq 0$

inequality!

Constraints introduce two types of difficulties in any mechanical system.

(i) The co-ordinates $\vec{r_i}$'s are not all independent.

(ii) There are forces of constraints, (eg :- the force exerted by the wall on gas particles) are usually unknowns and we know only effect of those.

Not all of the equations in (A) are independent!

The first problem is circumvented by introducing generalized co-ordinates for holonomic systems. If there are K-equation of the form eq.(2), then there should be 3N - K independent co-ordinates or degrees of freedom. This now 3N - K d.o.f. can be expressed by a new set of independent variables $q_1, q_2, \ldots, q_{3N-K}$. These new co-ordinates are related to old co-ordinates as:

$$\vec{r}_1 = \vec{r}_1 (q_1, q_2, \ldots, q_{3N-K}, t)$$
$$\vdots$$
$$\vec{r}_N = \vec{r}_N (q_1, q_2, \ldots, q_{3N-K}, t)$$

These set of equations are also invertible. (assumption) and they satisfy K constraints also.

These generalized co-ordinates q_is may not be divided into groups of three that can be associated together to form vectors. Eg:- particle confined to move on surface of a sphere, the latitude and longitude (ϕ, θ) are the generalized co-ordinates.

For non-holonomic systems (in few cases) one can employ Lagrange's multiplier method to include constraints equations along with the eqns of motion and solve the problem. → Lagrange's eqn. of motion of 1st kind.

The second difficulty, namely the forces of constraint can be handled by formulating the system in which forces of constraint disappear. We then only need to concentrate on applied forces. Hint:- Internal forces dont work for a rigid body!

D' Alembert's principle.

~~Let a system is imposed by d~~

Let m holonomic constraints (finite) are imposed on a system.

$$f_K(t, \vec{r}_i) = 0, \quad (K=1, 2, \ldots, m)$$

Now,
$$\sum_{i=1}^{N} \underbrace{\frac{\partial f_K}{\partial \vec{r}_i} \vec{v}_i}_{\vec{\nabla}_i f \cdot \vec{v}_i} + \frac{\partial f_K}{\partial t} = 0, \quad (K=1,2,\ldots,m) \quad \text{(i)}$$

The system of vectors $\vec{v}_i$ will be called possible velocities at a certain instant time t and for a certain possible position (at that instant) if $\vec{v}_i$'s satisfy Eqs (i). In general there can be further 'n' no of differential constraints present in the system satisfying:-

$$\cancel{\sum \vec{l}_i \vec{v}_i} \quad \sum_{i=1}^{N} \vec{l}_{pi} \vec{v}_i + D_p = 0 \ldots \text{(ii)}$$
$$(p=1,2,\ldots,n)$$

So possible velocities should satisfy m + n no of equations. Possible velocities are those which are permitted by the constraints.

Possible displacements :- $d\vec{r}_i = \vec{v}_i \, dt \quad (i = 1, 2, \dots N)$ (iii)

displacements consistent with the constraints. (forces, constraints may be changing during dt).

Now (i) $\Rightarrow$

$$\sum_{i=1}^{N} \frac{\partial f_k}{\partial \vec{r}_i} d\vec{r}_i + \frac{\partial f_k}{\partial t} dt = 0 , \quad (k = 1, 2, \dots, m) \quad (iv)$$

Let us take two system of possible displacements at one and the <u>same instant</u> of time and one and same position of the system.

$$d\vec{r}_i = \vec{v}_i \, dt \quad \text{and} \quad d\vec{r}_i{}' = \vec{v}_i{}' \, dt$$

$$\delta \vec{r}_i = d'\vec{r}_i - d\vec{r}_i \quad (v)$$

now eqn. (iv) $\Rightarrow \displaystyle\sum_{i=1}^{N} \frac{\partial f_k}{\partial \vec{r}_i} \delta \vec{r}_i = 0 \quad \dots (vi) \quad (k = 1, 2, \dots m)$

$\delta \vec{r}_i \equiv$ Virtual displacement

Any system of vectors $\delta \vec{r}_i$ satisfying (vi) is a system of virtual displacements.

In the case of stationary constraints, virtual displacements co-incide with possible displacements.