Classical Mechanics

IIIT, Allahabad

Instructor: Srijit Bhattacharjee

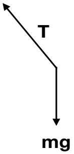
Constraints and Generalized coordinates

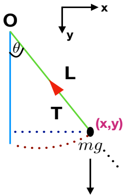
Constraints

Simple pendulum

$$m\ddot{x} = -T\sin\theta$$

$$m\ddot{y} = mg - T\cos\theta$$





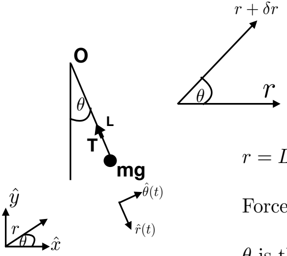
$$\tau = I\alpha \implies \ddot{\theta} = -\frac{g}{L}\sin\theta$$

$$r = L$$

$$x^2 + y^2 = L^2$$

Use of cartesian coordinates is not suitable. Polar coordinates are ideal.

Polar coordinates



$r = L$ so only θ changes. The # degree of freedom is only one.

Force equation: $mL\ddot{\theta} = -mg\sin\theta$

θ is the generalized coordinate. No EOM for r!

T is force of constraints.

Using $x = L\sin\theta$; $y = L\cos\theta$
One can show

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}; \quad \hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y}$$

And for SP

$$v = L\dot{\theta}\hat{\theta}$$

Find out v for general case i.e. for r varying.

A new approach

Lagrangian for Simple Pendulum

Lagrangian L is a function of generalised coordinate and generalised velocity.

$L = T - V$

$$L(\theta, \dot{\theta}) = \frac{1}{2} m L^2 \dot{\theta}^2 - mgL(1 - \cos\theta)$$

Lagrange's EOM:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

yields

$$mL^2 \ddot{\theta} = -mgL \sin\theta$$

- Lagrange's EOM is nothing but Newton's EOM.
- For a simple pendulum we have already seen that.
- Lagrangian of a particle of mass m attached to a spring with spring constant k and executing simple harmonic motion in x direction:

$$L(x, \dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

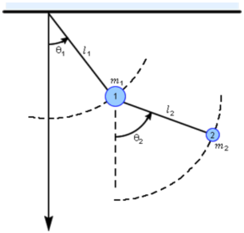
- We know the EOM from Newtonian mechanics: $m\ddot{x} = -kx$
- Lagrangian method:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}; \ \frac{\partial L}{\partial x} = -kx$$

$$\implies m\ddot{x} = -kx$$

Double Pendulum

Difficult to solve by Newtonian mechanics. Much easier in Lagrangian mechanics



Takeaways from the examples discussed

- Mechanical systems usually contain constraint forces.
  Eg. i) Tension on the bob of the simple pendulum. ii) Normal force exerted on a particle confined to move on a spherical surface.
- Systems with many degrees of freedom are hard to analyze in the Newtonian way. It is not easy to know the positions and velocities of all particles at a given moment.
  Eg. gas molecules in a box.
- Constraints impose two types of difficulties in a physical system.
  - Not all the coordinates describing the system (Cartesian coordinates) remain independent.
  - There are forces of constraints (Eg. force exerted on the gas molecules by the wall), that are generally unknown. We only used to know the effect of those.
- The first difficulty is removed by introducing the generalized coordinates.
- Removing the second difficulty (disappearance of forces of constraints) leads to the Lagrangian framework.

Generalized Coordinates

Generalized Coordinates: For a system of N particles, moving under constraints, the number of minimum independent coordinates $(q_1, q_2 \cdots q_n)$ required to describe the configuration (motion) of the system is called as system's *degrees of freedom* (dof) and the coordinates are called 'generalized coordinates' (GC).
There will be n generalized coordinates for a system with n dof. GC may not have the dimension of length.
For a system in D dimension, with k constraints,

$$n = D * N - k$$

Transformation Equations is the relation between position/Cartesian coordinates with GC , e.g. for $\vec{r_\nu} = x_\nu \hat{i} + y_\nu \hat{j} + z_\nu \hat{k}$

$$x_\nu = x_\nu(q_1, q_2, \cdots, q_n, t)$$
$$y_\nu = y_\nu(q_1, q_2, \cdots, q_n, t)$$
$$z_\nu = z_\nu(q_1, q_2, \cdots, q_n, t)$$

Classification of Mechanical systems

Scleronomic or Rheonomic: Rheonomic systems has explicit dependence on time t, but Scleronomic systems are not.

Eg. A particle is constrained to move on a stationary spherical surface of radius a (constraint eqn. $r - a = 0$) is an example of scleronomous system. If a particle is on a moving surface (eg. $f(t, x, y, z) = 0$) then we call that as a rheonomic system.

Holonomic or Non-holonomic: If constraints of a system can be expressed as $\phi(q_1, q_2, \cdots, q_n, t) = 0$ then the system is holonomic, else non-holonomic (eg. a particle is sliding on the surface of a sphere of radius a, $r^2 - a^2 \geq 0$).

Conservative systems: If all forces acting on a system are derivable from a potential $\left(F_x = -\frac{\partial V(x)}{\partial x}\right)$, then the system is conservative.

Virtual Displacement

- Let on a system of N particles, m holonomic constraints of the form $f_k(t, \mathrm{r}_i) = 0$ are imposed, then

$$\sum_{i=1}^{N} \frac{\partial f_k}{\partial \mathrm{r}_i} \mathrm{v}_i + \frac{\partial f_k}{\partial t} = 0; \;\; k = 1, \cdots, m. \tag{1}$$

- **Possible velocities and displacements:** The velocities satisfied by Eq. 1 are the possible velocities of the system at an instant t and at any specific position. The corresponding displacements $d\mathrm{r}_i = \mathrm{v}_i dt$, are the possible displacements. They are consistent with the constraints, but the forces of constraints may change over the interval dt.
- **Virtual displacements:** Let us consider two possible displacements at the same instant and the same position of the system at that instant.

$$d\mathrm{r}_i = \mathrm{v}_i dt \;;\;\; d'\mathrm{r}_i = \mathrm{v}'_i dt$$

plugging these into Eq. 1 and subtracting we get-

$$\sum_{i=1}^{N} \frac{\partial f_k}{\partial \mathrm{r}_i} \delta \mathrm{r}_i = 0; \;\; k = 1, \cdots, m. \tag{2}$$

Virtual Displacement & Principle of virtual work

- $\delta \mathrm{r}_i = d'\mathrm{r}_i - d\mathrm{r}_i$ is called the *virtual displacement*.
- Virtual displacements are also consistent with the constraints imposed on the system but they only coincide with the possible displacements in the case of stationary (time-independent) constraints.
- If the system is at equilibrium then the total force on each particle of the system must vanish.

$$\mathrm{F_i} = \mathrm{F_i^a} + \mathrm{F_i^c} = 0$$

- Total work done by the virtual displacement $\delta \mathrm{r_i}$ also vanishes.

$$\sum_{i=1}^{N} \mathrm{F_i}.\delta \mathrm{r}_i = 0 \implies \sum_{i=1}^{N} \mathrm{F_i^a}.\delta \mathrm{r}_i + \mathrm{F_i^c}.\delta \mathrm{r}_i = 0$$

- The net virtual work done by the forces of constraints usually vanishes (eg. internal forces on rigid bodies). Therefore

$$\sum_{i=1}^{N} \mathrm{F_i^a}.\delta \mathrm{r}_i = 0 \tag{3}$$

Eq. (3) is known as the Principle of Virtual Work.

D'Alembert's Principle

- The Principle of virtual work is only applicable to the study of statics. For general motion, D' Alembert's principle states- a particle maintains an equilibrium upon the joint action of the net force acting upon it F_i and a 'reverse effective force' $-\dot{\mathrm{p}}_i$. This follows from the Newton's law: $\mathrm{F}_i = \dot{\mathrm{p}}_i$.
- General Equation of dynamics: For the systems where the virtual work done by the constraint forces vanishes, we have:-

$$\sum_{i=1}^{N} ({\mathrm{F}^a}_i - \dot{\mathrm{p}}_i).\delta \mathrm{r}_i = 0 \tag{4}$$

- $\dot{\mathrm{p}}_i = ma_i$ is called *inertial force*.
- Eq. (4) is known as *D'Alembert's Principle*. It is still in terms of the Cartesian coordinates, which are not independent. One needs to transform to the generalized coordinates.
- Relation between the Cartesian and $n(= 3N - m)$ GC:

$$\mathrm{r}_i = \mathrm{r}_i(q_1, q_2, \cdots, q_n, t) \qquad N \text{ such relations.}$$

Lagrange's Equation

- $\delta \mathrm{r}_i = \sum_{\alpha=1}^{n} \frac{\partial \mathrm{r}_i}{\partial q_\alpha} \delta q_\alpha$
- $\sum_{i=1}^{N} F_i^a . \delta \mathrm{r}_i = \sum_{k=1}^{n} \Phi_k \delta q_k$.
- $\Phi_k = \sum_{i=1}^{N} F_i \cdot \frac{\partial r_i}{\partial q_k}$ is called *generalized force* associated with *generalized coordinate* q_k.
-

$$\sum_{i=1}^{N} \dot{p}_i . \delta \mathrm{r}_i = \sum_{i=1}^{N} m_i \ddot{\mathrm{r}}_i . \delta \mathrm{r}_i = \sum_{k=1}^{n} \sum_{i=1}^{N} \left[\frac{d}{dt} \left(m_i \mathrm{v}_i . \frac{\partial \mathrm{v}_i}{\partial \dot{q}_k} \right) - m_i \mathrm{v}_i . \frac{\partial \mathrm{v}_i}{\partial q_k} \right] \delta q_k$$

$$\implies \sum_{k=1}^{n} \sum_{i=1}^{N} \left[\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_k} (\tfrac{1}{2} m_i v_i^2) \right) - \frac{\partial}{\partial q_k} (\tfrac{1}{2} m_i v_i^2) \right] \delta q_k$$

- Eq. (4) becomes:

$$\sum_{k=1}^{n} \left[\Phi_k - \left(\sum_{i=1}^{N} \frac{d}{dt} \frac{\partial}{\partial \dot{q}_k} \left(\frac{1}{2} m_i v_i^2 \right) - \frac{\partial}{\partial q_k} \left(\frac{1}{2} m_i v_i^2 \right) \right) \right] \delta q_k = 0$$

Lagrange's Equation

For holonomic constraints, we can apply a virtual displacement in kth generalized coordinate q_k keeping other coordinates fixed. This gives:

$$\left[\Phi_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right], \tag{5}$$

where $T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$ is the total Kinetic energy.

If the force is derivable from a potential V-

$$\Phi_k = -\sum_{i}^{N} \nabla_i V \cdot \frac{\partial r_i}{\partial q_k} = -\frac{\partial V}{\partial q_k}$$

Plugging this in Eq. (5) we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0; \;\; L = T - V; \; k = 1, \cdots, n$$

Algorithm to obtain the Lagrangian and EOMs

1. Identify a suitable coordinate system that can serve as the GC.

2. Construct the kinetic and potential energy in terms of generalized coordinates and their derivatives. Then construct the Lagrangian $L = L(q_k, \dot{q}_k, t) = T - V$. Lagrangian is a function of generalized coordinates, generalized velocities and time.

3. Evaluate Lagrange's equations of motion by taking derivatives of the Lagrangian w.r.t. the GCs and GVs. There will be n number of equations for n number of GCs.

Force of constraints (extra reading)

- Consider the Lagrangian of simple pendulum without imposing the constraint $r = L$:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta - V(r).$$

- $V(r)$ can also be introduced as $\lambda(r - L)$, where λ is called the Lagrange's undetermined multiplier.
- r-equation:

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = -V^{'}(r) = -\lambda$$

- putting $r = L$:

$$\lambda - mg\cos\theta = mL\dot{\theta}^2$$

- λ is nothing but the constraint force T or tension!

Advantages of Lagrangian formulation

- Lagrangian is a scalar quantity. It is always advantageous to deal with scalars rather than vectors.
- Forces of constraints are dealt with elegantly. One can bypass the constraints and find the EOMs satisfying the constraints of a system.
- Lagrange's EOM can be obtained from a strong and elegant principle- The variational principle.