Classical Mechanics Tutorial

Engineering Physics

Indian Institute of Information Technology, Allahabad

Calculus of Multivariable Functions

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Now consider an example of total derivative. Let us consider a function $f(x(t), y(t), t)$ as

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$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t} = f_x\dot{x} + f_y\dot{y} + f_t \tag{6}$$

now you can calculate partial derivatives of f, i.e., f_x, f_y and f_z.

$$f_x = 2xyt + y^2 t \quad ; \quad f_y = x^2 t + 2xyt + t^2 \quad ; \quad f_t = x^2 y + xy^2 + 2yt \tag{7}$$

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If you divide by dt, you can will recover the Eq.(6).

- Now let us look at an example of double partial derivative. Consider a function $f(x, y) = x^2y + xy^2$. Calculate $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ and $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$?

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Calculate $f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and check if they are equal?

Constraints

Holonomic constraints – They can be expressed as an equation connecting the cooridnates of the particles. Eg. $f(t, x, y, z) = 0$
Non-holonomic constraints – Constraints which are not expressible in the above form of an equation.
Before moving on, recall that constraints can also be rheonomous (explicit time dependence) or scleronomous (not explicitly dependent of on time)

Example of constraints

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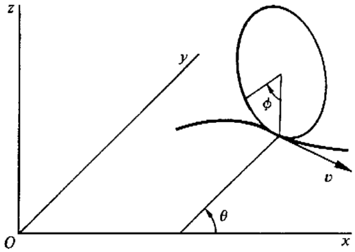
Another example of a non-holonomic constraint

Consider a disc of radius R rolling (without slipping) on a horizontal plane $x - y$ plane constrained to move so that the plane of the disc is always vertical. To describe the motion of the disc, we use the following coordinates: x, y coordinates of the center of the disc, the angle θ bewteen the axis of the disc and the x axis, and the angle of rotation ϕ about the axis of the disc.



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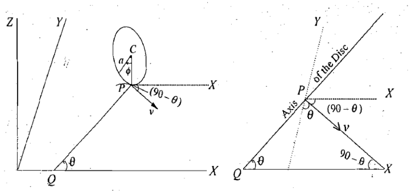

Since the disc remains vertical, the axis of rotation is perpendicular to the z axis. This tells us that the velocity of the center of the disc has a magnitude $|\mathrm{v}| = R\dot{\phi}$ and its direction is perpendicular to the axis of rotation $\implies \dot{x} = v \sin\theta$ and $\dot{y} = -v\cos\theta$ which implies

$$dx - R\sin\theta d\phi = 0 \text{ and } dy + R\cos\theta d\phi = 0 \tag{15}$$

These constraints are not of the form $f(x, y, \theta, \phi) = 0$ and are hence non-holonomic. Actually neither of the equations can be integrated without solving the problem first, that is, we cannot first find the integrating factor $f(x, y, \theta, \phi) = 0$ that will covert them into exact differentials.

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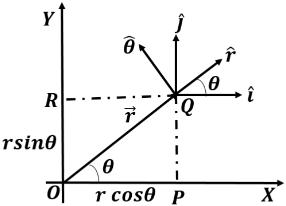

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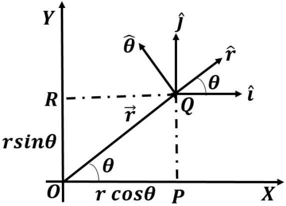

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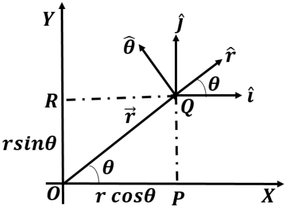

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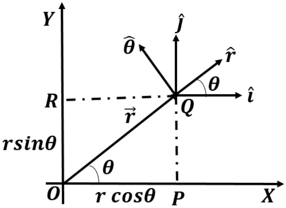

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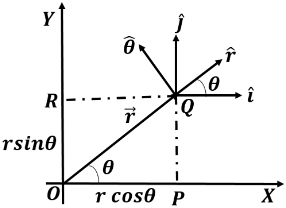

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- The Position vector is: $\vec{r} = x\hat{i} + y\hat{j} = r(\cos\theta\hat{i} + \sin\theta\hat{j})$

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Velocity construction:

- From the figure, we can write down the unit vectors for $\hat{r}$ and $\hat{\theta}$ in the direction of increasing θ and r respectively.

$$\hat{r} = \frac{\vec{r}}{|r|} = \cos\theta\hat{i} + \sin\theta\hat{j} \quad (17)$$

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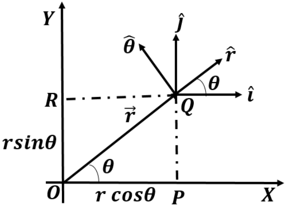

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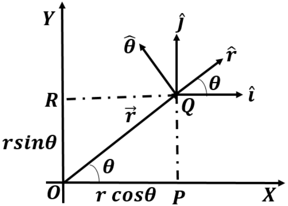

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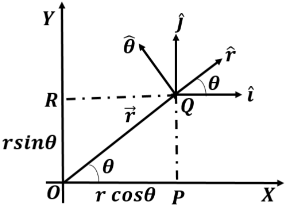

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$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \tag{21}$$

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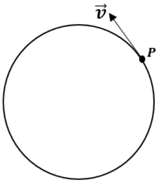
The term $\ddot{r}$ is *linear acceleration* in radial direction, $r\dot{\theta}^2$ is the *centripetal acceleration*, $\ddot{\theta}$ is the *acceleration* in the tangential direction, and $2\dot{r}\dot{\theta}$ is the *Coriolis acceleration*.

Example

Acceleration construction: Consider an object P moving on a circular path with a uniform velocity and radius R. Prove that it'll always be attracted towards the center of the circle.



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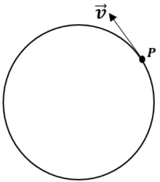

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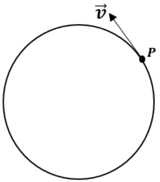

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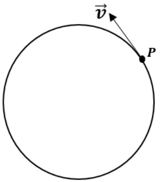

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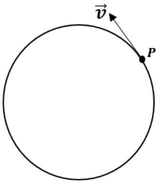

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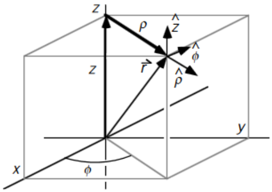

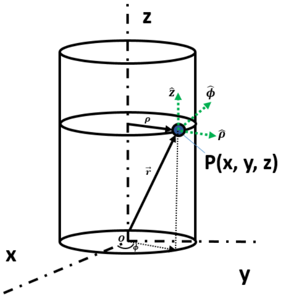

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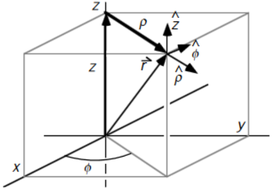

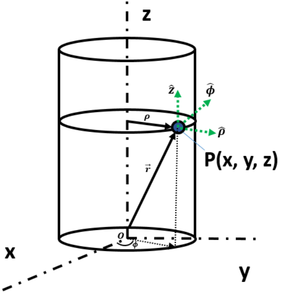

- The position vector for the point P is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{25}$$

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$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \tag{30}$$

$$\vec{a} = \dot{\vec{v}} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\phi} + \ddot{z}\hat{z} \tag{31}$$

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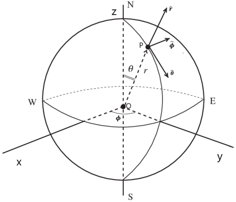

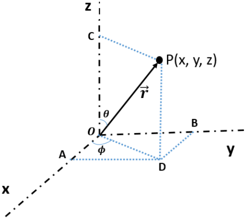

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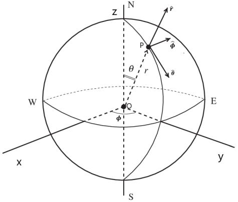

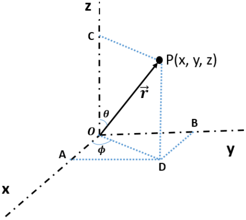

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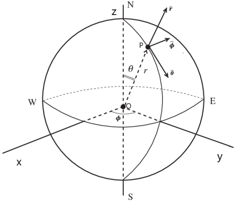

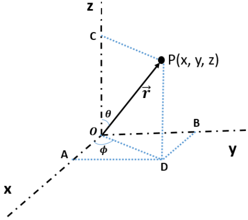

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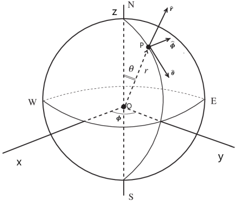

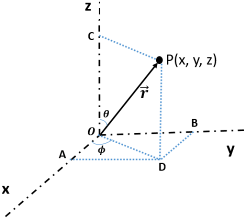

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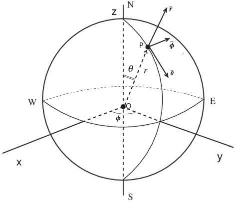

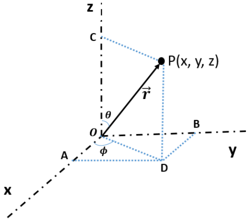

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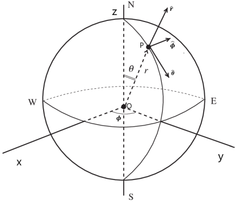

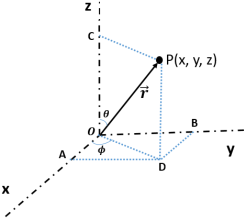

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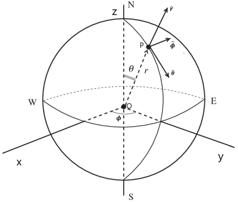

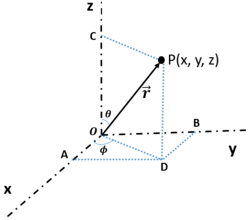

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- Find the expression for velocity & acceleration in spherical polar coordinate.