Classical Mechanics Tutorial II

Engineering Physics

Indian Institute of Information Technology, Allahabad

Invariance of Lagrangian under adding a total time derivative

Lagrange's EoM does not change by adding a total time derivative: $\frac{dF(q_1,q_2,\cdots,q_n,t)}{dt}$ Let, $L' = L + \frac{dF}{dt}$.
Since

$$\dot{F} = \frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t}$$

$\implies$

$$\frac{d}{dt}\left(\frac{\partial \dot{F}}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right)$$

$$\implies \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \left(\frac{\partial L'}{\partial q_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right)$$

Writing Down a Lagrangian

Consider a hydrogen atom consisting of a proton orbited by an electron at a fixed radius such that the electron is constrained to move on the surface of a sphere about the nucleus. What is the Lagrangian of this system?

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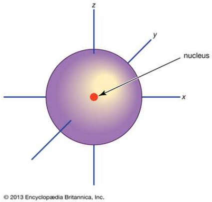
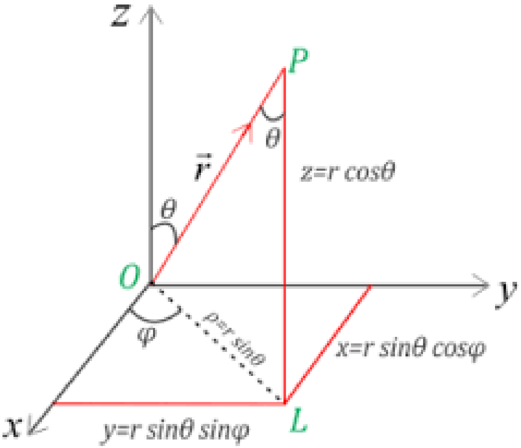
- We will be using spherical polar coordinates (r, θ, ϕ) to describe this system.
- The constraint is the fixed radius $r = l$, where l is an arbitrary constant indicating the fixed length.



Spherical Polar Coordinates



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We shall first obtain the expression of the kinetic energy T in spherical polar coordinates and then simply subtract the potential energy V from the kinetic energy to write down the Lagrangian.

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$$x = l \sin\theta \cos\phi \tag{1}$$

$$y = l \sin\theta \sin\phi \tag{2}$$

$$z = l \cos\theta \tag{3}$$

where $r = l$ from the constraint equation. The expression for kinetic energy T is simply given by

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{4}$$

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$$\dot{x} = l(\cos\theta \cos\phi\dot{\theta} + \sin\theta(-\sin\phi)\dot{\phi}) \tag{5}$$

$$\dot{y} = l(\cos\theta \sin\phi\dot{\theta} + \sin\theta(+\cos\phi)\dot{\phi}) \text{ and } \dot{z} = -l\sin\theta\dot{\theta} \tag{6}$$

So using these equations

$$\dot{x} = l(\cos\theta\cos\phi\dot{\theta} + \sin\theta(-\sin\phi)\dot{\phi}) \tag{7}$$

$$\dot{y} = l(\cos\theta\sin\phi\dot{\theta} + \sin\theta(+\cos\phi)\dot{\phi}) \tag{8}$$

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$$T = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin\theta^2\dot{\phi}^2) \tag{11}$$

We can now write the potential energy term as $V = V(\theta, \phi)$ and obtain the final Lagrangian $L = T - V$, that is,

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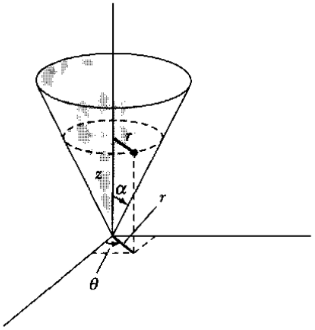
$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin\theta^2\dot{\phi}^2) - V(\theta, \phi) \tag{12}$$

Finding out the Equations of Motion

Let us consider a different problem: A particle of mass m is constrained to move on the inside surface of a smooth cone of half angle α. The particle is subject to a gravitational force. First determine a set of generalized coordinates and the constraints, and then find Lagrange's equations of motion.

Let us look at the figure:



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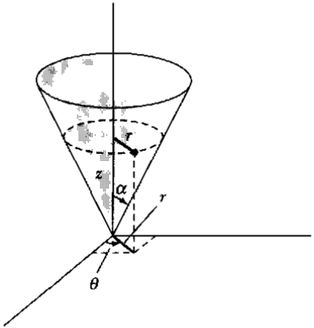

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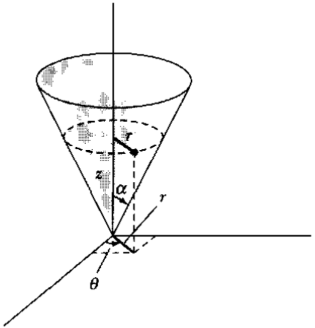

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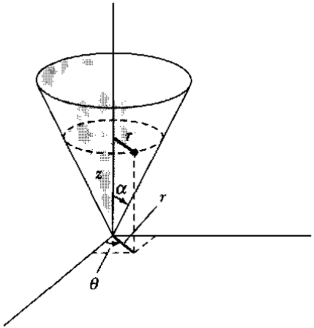

- We can see that using cylindrical polar coordinates (r, θ, z) will make the problem easier to solve.
- The constraint is the fixed radius $z = r \cot \alpha$, where we can see that r is the "height" and z is the "base" of the triangle formed by cutting the cone vertically. So $r/z = \tan \alpha$

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$$T = \frac{1}{2}m(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2) \tag{16}$$

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$$V = mgz = mgr\cot\alpha \tag{17}$$

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and,

$$V = mgz = mgr \cot \alpha$$

So the Lagrangian can be written as $L = T - U$

$$L = \frac{1}{2}m(\dot{r}^2 \csc^2 \alpha + r^2\dot{\theta}^2) - mgr \cot \alpha \tag{18}$$

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and Lagrange's equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0 \tag{19}$$

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \implies \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = const \tag{20}$$

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θ is a cyclic coordinate in this Lagrangian.

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But $mr^2\dot{\theta} = mr^2\omega$ is just the angular momentum about $z - axis$. This equation simply gives us the conservation of angular momentum. Similarly, we can calculate Lagrange's equation for the r coordinate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right) = 0 \tag{22}$$

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Computing the derivatives, we get

$$\ddot{r} - r\dot{\theta}^2 \sin^2 \alpha + g \sin \alpha \cos \alpha = 0 \tag{23}$$

Lagrange's equation of motion for dissipative systems

The Lagrange equation for a system with dissipation is given by

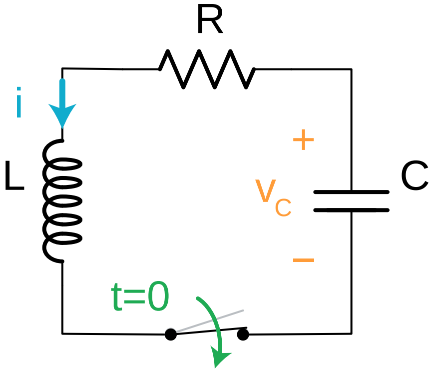
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0. \tag{24}$$

Here $\mathcal{F}$ is a scalar function known as Rayleigh's dissipation function and must be specified along with L to obtain the equations of motion.

One of the advantages of the Lagrangian formulation is that it can be easily extended to systems that are not studied in classical mechanics such as electrical circuits!

An example of a dissipative system: LCR circuits

We consider a physical system which has a "key" in series with an inductance L, A capacitance C and a resistance R. We first charge the capacitor and then close the key The capacitor will now begin to discharge. What is an equation that captures this situation?



An example of a dissipative system: LCR circuits

We choose the electric charge q as our dynamical variable (generalised coordinate).

The inductor will give rise to a kinetic energy term since the energy stored in an inductor contains current I, the time derivative of q. The inductance L is the electrical analogue of mass. So $= \frac{1}{2}L\dot{q}^2$.

The capacitor gives us the potential energy. The capacitance term $1/C$ is analogous to the spring constant k. So $U = \frac{1}{2}\frac{q^2}{C}$

An example of a dissipative system: LCR circuits

If we now introduce a dissipation function

$$\mathcal{F} = \frac{1}{2} R \dot{q}_j^2 \tag{25}$$

We can use

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0. \tag{26}$$

with

$$L = \frac{1}{2} L \dot{q}^2 - \frac{1}{2} \frac{q^2}{C} \tag{27}$$

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to get

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0 \tag{28}$$

This is exactly what we would get from Kirchhoff's law!

Taylor Series

Taylor's theorem provides a way of expressing a function as a power series in x, known as a Taylor series.

It can be applied only to those functions that are continuous and differentiable within the x-range of interest.

To express $f(x)$ as a power series in $x - a$ about the point $x = a$. We shall assume that, in a given x-range, $f(x)$ is a continuous, single-valued function of x having continuous derivatives. Then

$$f(x) = f(a) + (x-a)f^{'}(a) + \frac{(x-a)^2}{2!}f^{''}(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

alternatively setting $x = a + h$,

$$f(a+h) = f(a) + hf^{'}(a) + \frac{h^2}{2!}f^{''}(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots$$

Example: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}$.

Taylor Series: More than one variable: The Taylor series work the same way for functions of two variables. (There are just more of each derivative!) For an analytic function $f(x, y, z)$,

$$\begin{aligned} f(x,y,z) \approx & f(a,b) + f_x(a,b,z)(x-a) + f_y(a,b,z)(y-b) \\ & + \frac{f_{xx}(a,b,z)}{2}(x-a)^2 + f_{xy}(a,b,z)(x-a)(y-b) + \frac{f_{yy}(a,b,z)}{2}(y-b)^2 + \cdots \end{aligned} \quad (29)$$

Or, with $x = a + \delta x$ and $y = b + \delta y$

$$\begin{aligned} f(a+\delta x, b+\delta y, z) \approx & f(a,b,z) + f_x(a,b,z)\delta x + f_y(a,b,z)\delta y \\ & + \frac{f_{xx}(a,b,z)}{2}(\delta x)^2 + f_{xy}(a,b,z)\delta x \delta y + \frac{f_{yy}(a,b,z)}{2}(\delta y)^2 + \cdots \end{aligned} \quad (30)$$