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- In Hamiltonian framework we will transform the Lagrange's system of equations for n dof into $2n$ first order differential equations. 2 equations for each dof.
- Out of each set of two equations, one turns out to be again the Newton's eom. The other one usually gives the velocity of the system.

Legendre Transformation and Hamiltonian

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- From above we get,

$$g_x = -u; \; g_z = y.$$

Hamiltonian

- From $\mathcal{L}(q^\alpha, \dot{q}^\alpha, t)$ to $H(p_\alpha, q^\alpha, t)$ via Legendre Transformation:

$$H(p_\alpha, q^\alpha, t) = \sum_{\alpha=1}^{n} p_\alpha \dot{q}^\alpha - \mathcal{L}$$

With

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}.$$

-

$$d\mathcal{L} = \sum_{\alpha=1}^{n} \left(\frac{\partial \mathcal{L}}{\partial q^\alpha} dq_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} d\dot{q}^\alpha \right) + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$dH = -d\mathcal{L} + \sum_{\alpha=1}^{n} (dp_\alpha \dot{q}^\alpha + p_\alpha d\dot{q}^\alpha)$$

p,q are called canonically conjugate variables.

Hamilton's equation of motion

Using $dH = \sum_{\alpha=1}^{n} \left(\frac{\partial H}{\partial q^\alpha} dq^\alpha + \frac{\partial H}{\partial p_\alpha} dp_\alpha \right) + \frac{\partial H}{\partial t} dt$ we get,

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Example

- Let us consider the Lagrangian of a central force motion (planetary motion).

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EOMs

- $r-$EOM: $\frac{\partial H}{\partial r} = -\dot{p}_r = -\frac{p_\theta^2}{mr^3} + V'(r)$; $\frac{\partial H}{\partial p_r} = \dot{r} = p_r/m$.

- $\theta-$EOM: $r^2\dot{\theta} = p_\theta/m$, $\dot{p}_\theta = 0$, $\implies p_\theta = mr^2\dot{\theta} = const.$

- θ is absent in the Hamiltonian. The momentum conjugate to θ is conserved.

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- For holonomic, conservative systems, Hamiltonian is a measure of total energy of the system.

$$H = T + V$$

Cyclic Coordinates

cyclic coordinate: If q^α does not appear explicitly in the Lagrangian. In this case,

$$\dot{p}_\alpha = \frac{\partial L}{\partial q^\alpha} = 0$$

and p_α is called *constant of motion*. p_α is conserved.

We also have $\frac{\partial H}{\partial q^\alpha} = 0$

For conservative systems, time derivative of H becomes zero so total energy is conserved.

Summary

- Systems with many dof and constraints can be studied in Lagrangian as well as Hamiltonian framework.
- Lagrange's and Hamilton's equation of motions can be derived from variational principle.
- Hamiltonian of a system can be obtained from Lagrangian via Legendre mapping.
- Hamiltonian is dependent on canonical pairs -the gen. coordinates and gen. momenta. They constitute the phase space of the system.
- Hamiltonian is a measure of total energy of a system. In quantum mechanics it plays crucial role to figure out the dynamics of a system. Its *eigenvalues* provide the various energy states of a quantum system.