Classical Mechanics Tutorial

Engineering Physics

Indian Institute of Information Technology, Allahabad

Writing Down a Hamiltonian

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- We shall first obtain the Lagrangian of the system.
- We will then find the generalized momenta and express $\dot{q}$ in terms of p
- Finally we shall use $H = \sum p\dot{q} - L$ to obtain the Hamiltonian

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Lastly, since $p = \partial L/\partial \dot{x} = m\dot{x} \implies \dot{x} = p/m$, We get

$$T = \frac{p^2}{2m} \tag{3}$$

So using these equations

$$H = \sum p\dot{x} - L \tag{4}$$

$$= p\frac{p}{m} - \frac{p^2}{2m} + \frac{1}{2}kx^2 + \frac{1}{4}bx^4 \tag{5}$$

where

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We can now figure out the equations of motion.

Finding out the Equations of Motion

We have found out that

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 + \frac{1}{4}bx^4 \tag{8}$$

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Now we can differentiate 10 and use 11 to write
$m\ddot{x} = \dot{p} \implies m\ddot{x} + kx + bx^3 = 0$

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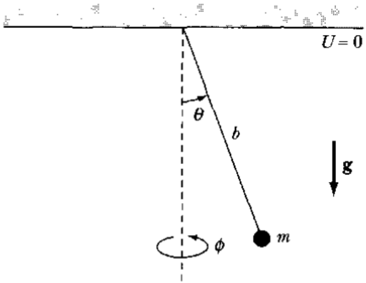

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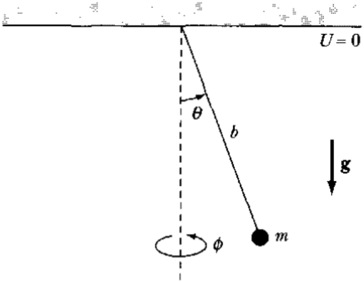

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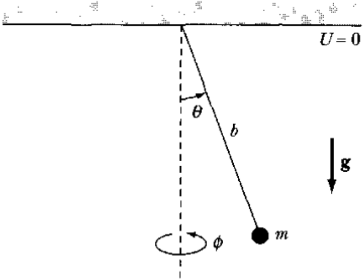

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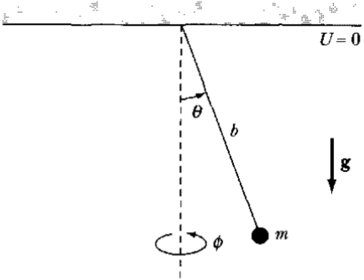

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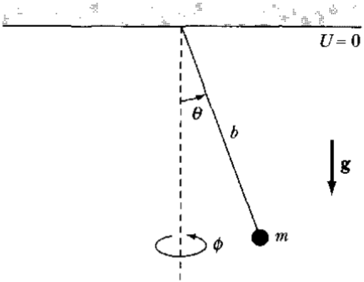

The kinetic energy is $T = \frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}mb^2\sin^2\theta\dot{\phi}^2$ and the potential energy is $U = -mgb\cos\theta$

Finding out the Equations of Motion

We can now write the Lagrangian as

$$L = \frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}mb^2\sin^2\theta\dot{\phi}^2 + mgb\cos\theta \tag{12}$$

and calculate the generalized momenta as

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$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mb^2\dot{\theta} \tag{13}$$

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We then calculate the Hamiltonian using $H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

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$$\begin{aligned} H = {} & p_\theta\frac{p_\theta}{mb^2} + p_\phi\frac{p_\phi}{mb^2\sin^2\theta} - \frac{1}{2}mb^2\left(\frac{p_\theta}{mb^2}\right)^2 \\ & - \frac{1}{2}mb^2\sin^2\theta\left(\frac{p_\phi}{mb^2\sin^2\theta}\right)^2 - mgb\cos\theta \end{aligned}$$

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we can write $H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ as

$$H = p_\theta\frac{p_\theta}{mb^2} + p_\phi\frac{p_\phi}{mb^2\sin^2\theta} - \frac{1}{2}mb^2\left(\frac{p_\theta}{mb^2}\right)^2 - \frac{1}{2}mb^2\sin^2\theta\left(\frac{p_\phi}{mb^2\sin^2\theta}\right)^2 - mgb\cos\theta$$

$$H = \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} - mgb\cos\theta \tag{15}$$

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$$\dot{p_\phi} = -\frac{\partial H}{\partial \phi} = 0 \tag{19}$$

Because ϕ is a cyclic coordinate, the momentum p_ϕ about the symmetry axis is constant