Classical Mechanics

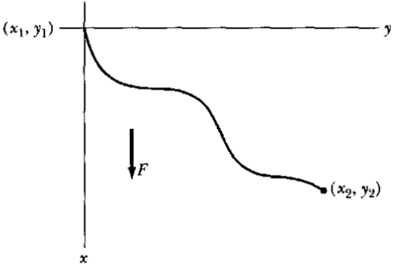
Calculus of Variation

Indian Institute of Information Technology, Allahabad

Calculus of Variations

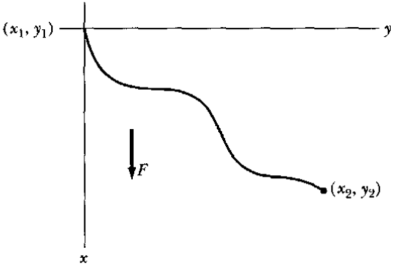
Let us first solve a classic problem from the history of physics, the brachistochrome, using the calculus of variations. Consider a particle moving in a constant force field starting from rest from some point (x_1, y_1) to some lower point (x_2, y_2). We have to find the path that allows the particle to accomplish this transit in the least possible time

We chose the coordinate system such that the point (x_1, y_1) is at the origin



Calculus of Variations

The force F is along the positive x direction. Because the force on the particle is constant, and if we ignore the possibility of friction, the field is conservative and total energy is constant. We also consider the particle to be initially at rest, and $V = 0$ at $x = 0$:



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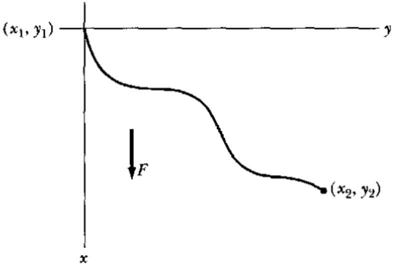

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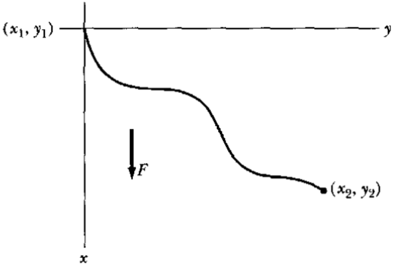

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So the time required for the particle to make a transit from origin to (x_2, y_2) is

$$t = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{v} = \int \frac{(dx^2 + dy^2)^{1/2}}{(2gx)^{1/2}} \tag{1}$$

$$t = \int_{x_1=0}^{x_2} \left(\frac{1 + y'^2}{2gx} \right)^{1/2} dx \tag{2}$$

The time of transit is to be minimized , and since the constant $(2g)^{-1/2}$ does not effect the final equation, we identify L as

$$L = \left(\frac{1 + y'^2}{x} \right)^{(1/2)} \tag{3}$$

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This gives us

$$\frac{y'^2}{x(1+y'^2)} = \frac{1}{2a} \tag{6}$$

which may be written as

$$y = \int \frac{xdx}{(2ax - x^2)^{1/2}} \tag{7}$$

Calculus of Variations

We can solve this integration

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by introducing the following change of variable

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We then the get the parametric equations of the cycloid

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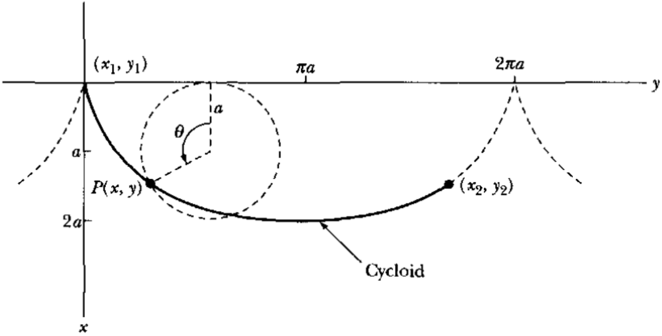

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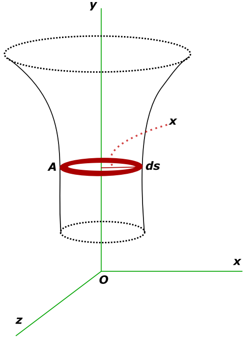
- Consider a curve between two points (x_1, y_1) and (x_2, y_2) in the xy plane whose equ is $y = y(x)$. We form a surface by revolving the curve about y-axis. We are interested in finding the nature of the curve for which the surface area is minimum.



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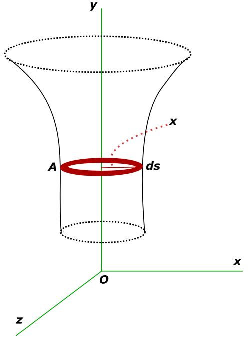

Consider a small strip at a point A formed by revolving the arc length ds about y–axis.

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The total area of the surface of revolution of the curve $y = y(x)$ about y- axis is given by

$$A = \int_{x_1}^{x_2} 2\pi x\sqrt{1 + y'^2}dx \tag{17}$$

Calculus of Variation

This surface area will be minimum iff the integrand $f = 2\pi x\sqrt{1 + y'^2}$ satisfies Euler-Lagrange's equation, i.e.

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Solving for y' gives

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}}. \tag{20}$$

Integration of this will result into

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The above equation is known as the Catenary Curve.