# Quantum Gates (Single Qubit)

## Circuit diagrams

## Pauli X, Y, Z gates

$$X \equiv \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad -\boxed{X}- \equiv -\otimes-$$

(NOT gate)

$$Y \equiv \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad -\boxed{Y}-$$

$$Z \equiv \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad -\boxed{Z}-$$

$H \equiv$ Hadamard gate $\equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad -\boxed{H}-$

Phase gate $\equiv \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad -\boxed{S}-$

$\frac{\pi}{8}$ Gate or T Gate $\equiv \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \equiv e^{i\pi/8}\begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} \qquad -\boxed{T}-$

Operations :- Eg:

$$|0\rangle -\boxed{X}- |1\rangle$$

$$|1\rangle -\boxed{X}- |0\rangle$$

# Multi Qubit Gates

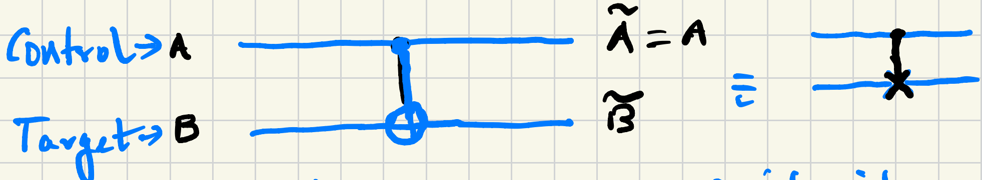
## C-Not Gate / Controlled NOT Gate.

C-NOT gate is a universal gate.

Like NAND gate, which is a universal gate in the sense, any boolean operation can be performed using a NAND gate w/o using any other basic logic gates.

C-NOT gate is a 2-qubit gate.

It has a control qubit (1st qubit) and a target qubit




This gate produces a $|0\rangle$ if it acts on $|0\rangle$ but flips the input if it acts on $|1\rangle$

$\therefore$

$$\text{CNOT}\,|00\rangle = |00\rangle$$

$$\text{CNOT}\,|01\rangle = |01\rangle$$

$$\text{CNOT}\,|10\rangle = |11\rangle$$

$$\text{CNOT}\,|11\rangle = |10\rangle$$

This can be understood as $\rightarrow$ $\text{CNOT}\,|a\rangle \otimes |b\rangle = |a\rangle \otimes (|b\rangle \oplus |a\rangle)$

where $\oplus$ means addition modulo 2.

CNOT can generate entanglement

$$\text{CNOT}\,(a|0\rangle + b|1\rangle) \otimes |0\rangle$$

$$= a|00\rangle + b|11\rangle$$

**Matrix representation:-**

$$\langle 00|\,\text{CNOT}\,|00\rangle = 1$$

$$\langle 01|\,\text{CNOT}\,|01\rangle = 1$$

$$\langle 10|\,\text{CNOT}\,|11\rangle = 1$$

$$\langle 10|\,\text{CNOT}\,|11\rangle = 1$$

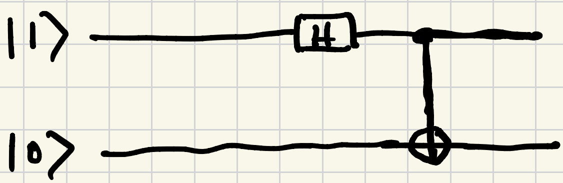
In the $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ basis
$\quad\quad\quad\quad |1\rangle \quad |2\rangle \quad |3\rangle \quad |4\rangle$

$$CNOT \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow U_{CNOT} \doteq \left(|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 4| + |4\rangle\langle 3|\right)$$

Check: $CNOT^2 = \mathbb{I}$

CNOT gate is used to produce an entangled state.




$$\frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

w/o CNOT:- $|1\rangle \xrightarrow{H} \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes |0\rangle$

$$\longrightarrow \frac{|00\rangle - |10\rangle}{\sqrt{2}}$$

Not entangled

But now use CNOT at the target bit and get $\frac{|00\rangle - |11\rangle}{\sqrt{2}}$

$\downarrow$

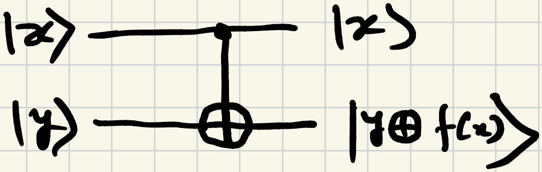
Entangled!

CNOT is like XOR gate

XOR is having same addition modulo 2 operation.

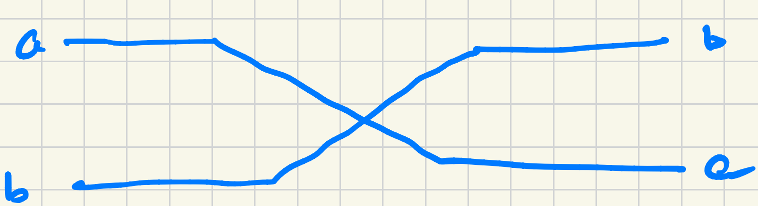
$$0 \oplus 0 = 0$$
$$0 \oplus 1 = 1$$
$$1 \oplus 0 = 1$$
$$1 \oplus 1 = 0$$

In general




## SWAP Gate

$$U_S \, |a\rangle \otimes |b\rangle = |b\rangle \otimes |a\rangle$$

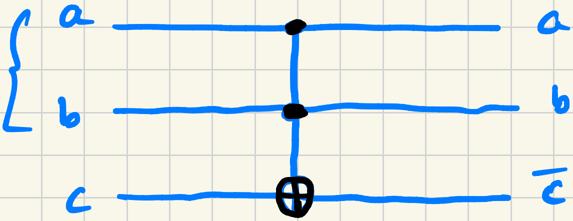



$$U_S \doteq |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|$$

# 3-Qubit Gate

## CCNOT / Toffoli Gate

two control bits




$\bar{c} = c \oplus ab$

8 × 8 Matrix

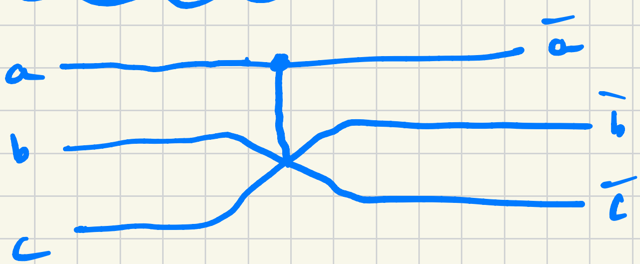
$$\left(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|\right) \otimes \mathbb{1} + \left(|11\rangle\langle 11|\right) \otimes X$$

for $c = 0$; $c$ is flipped iff $ab = 1$ i.e. $a = b = 1$ ≡ AND gate

o.w $\bar{c} = 0$

## C-SWAP gate

Fredkin gate




If control bit is 1, this gate interchanges two target bits.

# Quantum Circuits

Looping (like in classical case, several inputs having same output)

fan-in / fan-outs are not possible.

## Half-Adder Circuit

$1+0=1$

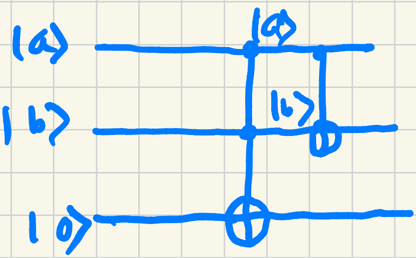
$0+0=0$

$0+1=1$

$1+1=10 \to$ sum

$\downarrow$

carry




$|b\rangle \oplus |a\rangle \to$ sum

$|0 \oplus ab\rangle \to$ CARRY

as $ab \neq 0$ iff $a = b = 1$

## Quantum Circuit components

Input, Gates, Oracle, Output

Measurement

Oracle is like subroutine in classical programs.

# Quantum Algorithms

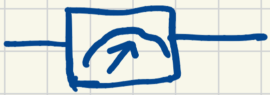
4- components

① Input :- Linear superposition of states.

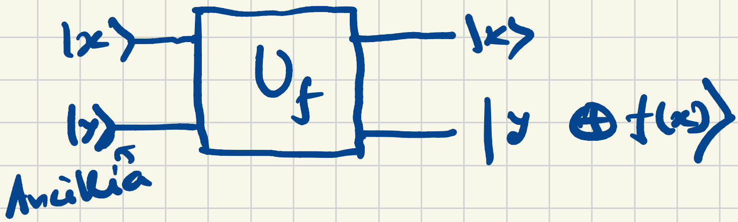
② Quantum parallelism :-

③ Oracle :- A black box

④ Measurement :-



③ Oracle :-




④ Measurement :- Only one possible state is measured with a certain probability. out of the all possible output states (in a linear superposition)

(Done usually in a computational basis).

# Simple Algorithms

## Deutsch Algorithm

Let $f : \{0, 1\} \longrightarrow \{0, 1\}$

It can be of two types

$$\left.\begin{array}{l} f(0) = 0 \\ f(1) = 0 \\ f(0) = 1 \\ f(1) = 0 \end{array}\right\} \text{Constant}$$

OR,

$$\left.\begin{array}{l} f(0) = 0 \ ; \ f(0) = 1 \\ f(1) = 1 \ ; \ f(1) = 0 \end{array}\right\} \quad \text{Balanced}$$

Can we design an algo to know that the fn. belongs to which type?

Classically this can be done by 2-queries if $f(0) = 0$

 if $f(1) = 0$

  print $\rightarrow$ constant

 else

  print $\rightarrow$ balanced

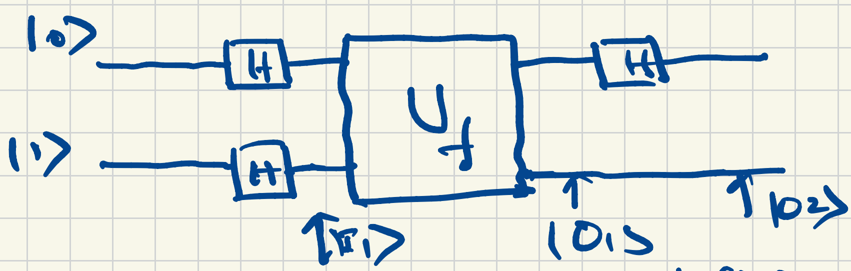
if $f(0) = 1$

 if $f(1) = 1$

  print $\rightarrow$ constant

 else print $\rightarrow$ balanced

Deutsch Algo can do it more efficiently



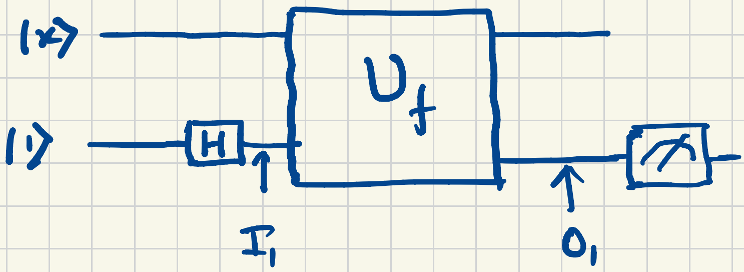

$|I_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$|O_1\rangle = \, ?$

If $f(0) = 0$

and $f(1) = 1$

An attempt to efficiently design a circuit.




$$|I_1\rangle = |x\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|x,0\rangle - |x,1\rangle\right)$$

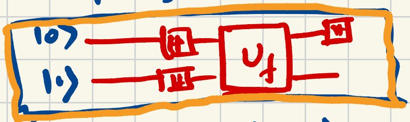
$$|O_1\rangle = \frac{1}{\sqrt{2}}\left(|x, 0 \oplus f(x)\rangle - |x, 1 \oplus f(x)\rangle\right)$$

$$= \begin{cases} \dfrac{|x,0\rangle - |x,1\rangle}{\sqrt{2}} & \text{if } f(x) = 0 \\ \dfrac{|x,1\rangle - |x,0\rangle}{\sqrt{2}} & \text{if } f(x) = 1 \end{cases}$$

Concrete way of writing the same :-

$$|O_1\rangle = (-1)^{f(x)} \frac{|x,0\rangle - |x,1\rangle}{\sqrt{2}}$$

$$|I_1\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$




$$|O_1\rangle = U_f\,|I_1\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle + |11\rangle - |10\rangle\right)$$

$$= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

If $f(0) = 1$ and $f(1) = 0$
this would have been a '−' sign

Now implementing a Hadamard (see pic) gate at top qubit $\Rightarrow$

$$\pm\ |1\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

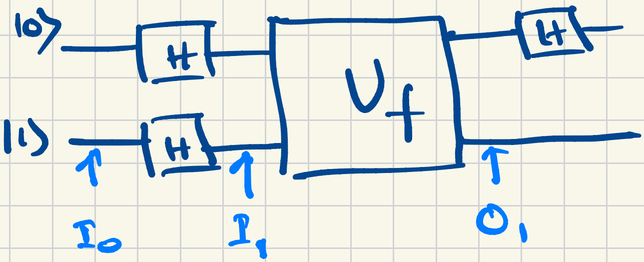
One can show for $f(0) = f(1)$
this will be

$$\pm\ |0\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

So, if one measures the 1st qubit then one determines the fn type i.e. constant or Balanced

Made with Goodnotes

We can also write :-




$$|I_0\rangle = |0\rangle \otimes |1\rangle$$

$$|I_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$U_f$ changes the top qubit, so

$$|O_1\rangle = \frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Now check for different f's.

Let, $f(0) = 0$ and $f(1) = 0$

$$\Rightarrow |O_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

and $f(0) = 1$ and $f(1) = 1$

$$\Rightarrow |O_1\rangle = -\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$\therefore$ for $f$ const

$$|0_1\rangle = \pm \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

also for balanced

$$|0_1\rangle = \frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$f(0) = 0$
$f(1) = 1$
or
$f(0) = 1$
$f(1) = 0$

$$\Rightarrow \pm \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$