Positive Operators

An operator A is positive semi-definite if for every $|v\rangle \in V$

$$\langle v|A|v\rangle \geq 0 \quad (> 0 \text{ for positive operator})$$

If $|v\rangle$ is an eigenvector of A with an eigenvalue λ

$$\langle v|A|v\rangle = \lambda \| v \|^2 \geq 0$$

So, all the λs must be positive / zero.

eigenvalues of a positive operator are positive.

Polar decomposition of an operator

If $\hat{A}$ is non-singular, then for U unitary

$$\hat{A} = \overset{\text{Right}}{U J} = \overset{\text{Left}}{K U}$$

where $J = \sqrt{\hat{A}\hat{A}^\dagger}$

and $K = \sqrt{\hat{A}^\dagger \hat{A}}$ are positive operators.

J and K are Hermitian

J has a spectral decomposition

$$J = \sum_i j_i |j_i\rangle\langle j_i|$$

with $j_i > 0$.

Let $\hat{A}|j_i\rangle = |\psi_i\rangle$

And, $U J |j_i\rangle = U j_i |j_i\rangle$

$$= j_i \underbrace{U|j_i\rangle}_{|e_i\rangle}$$

let $|e_i\rangle = \dfrac{|\psi_i\rangle}{j_i} = \dfrac{\hat{A}|j_i\rangle}{j_i}$

$|e_i\rangle \rightarrow$ normalised

$$\langle e_i | e_j\rangle = \frac{1}{j_i j_j} \langle j_i | \hat{A}^\dagger \hat{A} | j_j\rangle$$

$$= \frac{\delta_{ij}}{j_i j_j} j_j^2 = \delta_{ij}$$

$\{|e_i\rangle\} \rightarrow$ orthonormal basis

let $|e_i\rangle = U|j_i\rangle$

then

$$UJ|j_i\rangle = j_i U|j_i\rangle$$

$$= |\psi_i\rangle = \hat{A}|j_i\rangle$$

$\therefore \quad \hat{A} = UJ; \quad \hat{A}^\dagger = J^\dagger U^\dagger = JU^\dagger$

$$\hat{A}^\dagger \hat{A} = JU^\dagger UJ = J^2$$

$\therefore \boxed{J = \sqrt{\hat{A}^\dagger \hat{A}}}$ and $\boxed{U = AJ^{-1}}$

Ex:- Find polar decomposition of

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Ans. $A^tA = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$$= \begin{pmatrix} a^2+b^2 & 0 \\ 0 & b^2+a^2 \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \qquad r^2 = a^2+b^2$$

$$J = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = \sqrt{A^tA}$$

$$A = UJ \qquad U = AJ^{-1}$$

$$J^{-1} = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$$

$$U = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} J^{-1} = \begin{pmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{pmatrix}$$

For any Hermitian matrix A there can be a U unitary, so that

$$U^{\dagger} A U = A_d \rightarrow \text{diagonal}$$

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Singular Value Decomposition

For A $m \times n$

$$A = U \Sigma V^{\dagger}$$

$U \rightarrow$ unitary $m \times m$; V is unitary of $n \times n$

Σ is $m \times n$ matrix whose diagonal entries are positive semi definite

If A is real, U, V are orthogonal.

Proof:- $A = SJ \rightarrow$ polar decomposition

J is Hermitian and diagonalizable

$$J = T \overset{\text{non-negative}}{\Sigma} T^{\dagger} \rightarrow \text{unitary}$$

$\therefore$ $A = S T \Sigma T^{\dagger}$ define $\underbrace{ST}_{\text{both unitary}} = U$

$A = U \Sigma T^{\dagger}$ $T = V \rightarrow$ set

Show the product of two unitary operators is also unitary.

Diagonalization :-

Let an operator M can be represented w.r.t. a basis $\{|u_i\rangle\}$ $i=1,\dots,n$.

$$\langle u_i|M|u_j\rangle = M_{ij}$$

What is the operator M w.r.t. another basis $|v_i\rangle = S|u_i\rangle$?

$$\langle v_i|\,M'|\,v_j\rangle - \langle u_i|S^{-1}M\,S|u_j\rangle$$

Recall change of basis is done by Unitary operators

$$\therefore \quad M' = S^{-1}M\,S$$

M and M' are similar

If S is constructed by the eigenvectors of M i.e. if M has $|e_1\rangle, \dots |e_n\rangle$ eigenvectors

then
$$S \doteq \begin{pmatrix} |e_1\rangle & |e_2\rangle & \cdots & |e_n\rangle \\ & & & \end{pmatrix}$$

Then $M' = S^\dagger M S \doteq S^{-1} M S = M_d$

is diagonal

Proof:- Since

$$M|e_j\rangle = \lambda_j |e_j\rangle$$

$$M_{d\,jj} = \lambda_j \delta_{ij} \rightarrow \text{no sum}$$

Now, $S_{ij} \doteq |e_j\rangle^i$

So, $S M_d = \lambda_1 |e_1\rangle \oplus \cdots \oplus \lambda_n |e_n\rangle$

$$S \doteq \begin{pmatrix} \vdots & \vdots & & \vdots \\ |e_1\rangle & |e_2\rangle & \cdots & |e_n\rangle \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

$$M S = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \lambda_1 |e_1\rangle & \lambda_2 |e_2\rangle & \cdots & \lambda_n |e_n\rangle \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

$$= \lambda_1 |e_1\rangle \oplus \cdots \oplus \lambda_n |e_n\rangle$$

$\therefore \quad S M_d = M S \Rightarrow M_d = S^{-1} M S$

Check let $M \doteq \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$

$$|e_1\rangle \doteq \begin{pmatrix} e_1^{(1)} \\ e_2^{(1)} \end{pmatrix}$$

$$|e_2\rangle \doteq \begin{pmatrix} e_1^{(2)} \\ e_2^{(2)} \end{pmatrix}$$

$$\therefore \quad S = \begin{pmatrix} e_1^{(1)} & e_1^{(2)} \\ e_2^{(1)} & e_2^{(2)} \end{pmatrix}$$

$$\therefore \quad MS = \begin{pmatrix} M_{11}e_1^{(1)} + M_{12}e_2^{(1)} & M_{11}e_1^{(2)} + M_{12}e_2^{(2)} \\ M_{21}e_1^{(1)} + M_{22}e_2^{(1)} & M_{21}e_1^{(2)} + M_{22}e_2^{(2)} \end{pmatrix}$$

$$\Rightarrow \quad M|e_1\rangle \oplus M|e_2\rangle$$

$$\Rightarrow \quad \lambda_1|e_1\rangle \oplus \lambda_2|e_2\rangle$$

$$= S M_d$$

A $n \times n$ Matrix is diagonalizable iff the eigenvectors of it spans the space, i.e. if there exists n - linearly independent eigenvectors

Ex! - 1) Diagonalize the matrix

$$T = \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix}$$; construct the S s.t. $S^{-1} T S = T_d$

2) Show that the 2-d rotation matrix $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is diagonalizable.. Diagonalize it.

Expectation value of an operator example:-

Let, $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

What is $\langle 0 | H | 0 \rangle$? $\rightarrow$ expectation in state $|0\rangle$.

$\langle 0 | = (1 \ 0)$

$\therefore \quad (1 \ 0) \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \frac{1}{\sqrt{2}} (1 \ 0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \frac{1}{\sqrt{2}} (1+1) = \sqrt{2}$ Ans.

If $|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ then what is $\langle H \rangle$?

$\langle H \rangle = \langle \psi | H | \psi \rangle$

$= \frac{1}{2} (1 \ -1) \ H \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= \frac{1}{2\sqrt{2}} (1 \ -1) \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$= -\frac{1}{\sqrt{2}}$

Ex:
What is $\langle \psi | \sigma_x | \psi \rangle$?

$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$
What is $\langle \psi | \sigma_z | \psi \rangle$?
$= 1 \times \left|\frac{1}{\sqrt{2}}\right|^2 + (-1) \left|\frac{1}{\sqrt{2}}\right|^2$
$= 0$

Tensor Product

If $A \in \mathcal{H}_A$ and $B \in \mathcal{H}_B$ → Two Hilbert spaces

A ↓ system, B ↓ system

Then the composite system belongs in

$$\mathcal{H}_A \otimes \mathcal{H}_B$$

Given two vector spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ having orthonormal bases $\{|a_1\rangle, \ldots |a_n\rangle\}$ and $\{|b_1\rangle, \ldots, |b_m\rangle\}$, the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$ is spanned by

$$\{ |a_1\rangle \otimes |b_1\rangle, |a_1\rangle \otimes |b_2\rangle, \ldots |a_1\rangle \otimes |b_m\rangle;$$
$$|a_2\rangle \otimes |b_1\rangle, |a_2\rangle \otimes |b_2\rangle, \ldots |a_2\rangle \otimes |b_m\rangle;$$
$$\ldots\ldots |a_n\rangle \otimes |b_1\rangle, \ldots, |a_n\rangle \otimes |b_m\rangle \}$$

If $|a\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ two vectors $|a\rangle \in \mathcal{H}_A$

and, $|b\rangle = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ $|b\rangle \in \mathcal{H}_B$

$$|a\rangle \otimes |b\rangle \doteq |ab\rangle$$

$$|ab\rangle \doteq \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_1 b_3 \\ \vdots \\ a_1 b_m \\ \vdots \\ a_n b_1 \\ \vdots \\ a_n b_m \end{pmatrix}$$

$\underbrace{nm}_{\text{dim}} \times 1$ matrix

$$|0\rangle \otimes |1\rangle = |0\rangle|1\rangle = |01\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\alpha_1\rangle = a|0\rangle + b|1\rangle \; ; \; |\alpha_2\rangle = c|0\rangle + d|1\rangle$$

$$|\alpha_1 \alpha_2\rangle = ac|00\rangle + bd|11\rangle + ad|01\rangle + bc|10\rangle$$

This state $|\alpha_1 \alpha_2\rangle$ is a direct product of $|\alpha_1\rangle$ and $|\alpha_2\rangle$ states.

Not always an arbitrary two-qubit state can be decomposed into one-qubit states.

If $|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ → Bell state

this can't be decomposed into one-qubit state!
Entangled state!

A general 2-qubit state can be expressed as superposition of base states $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$.

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

with $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$

Ex:- Decompose $\frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$

Ans:- $\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

Properties of Tensor product spaces:-

If $|v_1\rangle \in V$, $|v_2\rangle \in V$ / $|w\rangle \in W$

then $\left(\alpha_1|v_1\rangle + \alpha_2|v_2\rangle\right) \otimes |w\rangle = \alpha_1|v_1 w\rangle + \alpha_2|v_2 w\rangle$

$\alpha_1, \alpha_2 \in \mathbb{C}$

Also

$$|v\rangle \otimes (\beta_1 |w_1\rangle + \beta_2 |w_2\rangle)$$
$$= \beta_1 |v w_1\rangle + \beta_2 |v w_2\rangle$$

Inner product

$$\langle v \otimes w | u \otimes x\rangle$$
$$= \langle v|u\rangle \langle w|x\rangle$$

Linear operators

If $\hat{A} \in L(V)$ and $\hat{B} \in L(W)$

then $(\hat{A} \otimes \hat{B})(|v\rangle \otimes |w\rangle)$

$$= \hat{A}|v\rangle \otimes \hat{B}|w\rangle$$

Measuring 2-qubit system means measuring 2^2 co-efficients. So 4 information. Measuring n-qubit system has 2^n information. Dimension of vector space is 2^n.

If $|\alpha\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

then probability of measuring both $|0\rangle, |0\rangle$ is $\Longrightarrow |\alpha_{00}|^2$

But probability of measuring 1st qubit to be 0 is

$$|\alpha_{00}|^2 + |\alpha_{01}|^2$$

Eg. Find

$$\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right) \otimes |-\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$\Rightarrow \frac{\sqrt{3}}{2\sqrt{2}}|00\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle - \frac{1}{2\sqrt{2}}|11\rangle$$

What is the probability that 1st qubit measured is a $|0\rangle$?

$$\left|\frac{\sqrt{3}}{2\sqrt{2}}\right|^2 + \left|\frac{\sqrt{3}}{2\sqrt{2}}\right|^2 = \frac{3}{4}$$

Eg:- $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ then

find $X_1 \otimes Z_2 |\psi\rangle$

1 → acts on 1st entry

2 → acts on 2nd entry

$$\Rightarrow \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)$$ Ans.

$\because$ $Z_2|1\rangle = \begin{pmatrix}0\\-1\end{pmatrix} = -|1\rangle$ $\quad Z_2|0\rangle \Rightarrow \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$

$X|1\rangle = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$ $\quad X_1|0\rangle \Rightarrow \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$

Quantum Gates (Single Qubit)

Circuit diagrams

Pauli X, Y, Z gates

$$X \equiv \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad -\boxed{X}- \equiv -\oplus-$$

(NOT gate)

$$Y \equiv \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad -\boxed{Y}-$$

$$Z \equiv \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad -\boxed{Z}-$$

$$H \equiv \text{Hadamard gate} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad -\boxed{H}-$$

$$\text{Phase gate} \equiv \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad -\boxed{S}-$$

$$\frac{\pi}{8} \text{ Gate or T Gate} \equiv \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \equiv e^{i\pi/8} \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} \qquad -\boxed{T}-$$

Operations :- Eg:

$$|0\rangle -\boxed{X}- |1\rangle$$

$$|1\rangle -\boxed{X}- |0\rangle$$