Quantum Computing and Information

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- Vector space, Bases, Operators

2-d examples :-

Standard basis

$$|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

'$\doteq$' Means Representing

$$|1\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Classical bits :- $\{0, 1\}$ anyone of these

Quantum bit / Qubit: $|1\rangle$ or $|0\rangle$ or linear combination of these

Ex:-

Suppose $|\psi\rangle \doteq \begin{pmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

A state vector

Represent $|\psi\rangle$ as a qubit.

if $|\psi\rangle = a|0\rangle + b|1\rangle$

find a, b.

$$|\psi\rangle \Rightarrow \frac{i}{\sqrt{2}}|0\rangle + \left(-\frac{1}{\sqrt{2}}\right)|1\rangle$$

$$a = \frac{i}{\sqrt{2}} \quad ; \quad b = -\frac{1}{\sqrt{2}} \quad \left\{ |a|^2 + |b|^2 = 1 \right.$$

$|0\rangle, |1\rangle$ forms a basis (orthonormal)

$\langle 0|0\rangle = 1$ show! $\langle 1|1\rangle = 1$

$\langle 0|1\rangle = 0$; $a = \langle 0|\psi\rangle$

$b = \langle 1|\psi\rangle$

Show $|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{I}_2$

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Projecting onto state $|0\rangle$

$$|0\rangle\langle 0|\psi\rangle$$

$$\Rightarrow \frac{i}{\sqrt{2}} \langle 0|0\rangle |0\rangle - \frac{1}{\sqrt{2}} \underbrace{\langle 0|1\rangle}_{=0} |0\rangle$$

$$\Rightarrow \frac{i}{\sqrt{2}} |0\rangle$$

Similarly $|1\rangle\langle 1|\psi\rangle = -\frac{1}{\sqrt{2}}|1\rangle$

If $\hat{A} = \hat{\mathbb{I}}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

then $|\psi\rangle$ can always be written as a linear superposition of $|0\rangle$ and $|1\rangle$ which are the bases of $\hat{A} = \hat{\mathbb{I}}$

Check!
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\hat{A}_1 \quad |0\rangle = |0\rangle$$
projecting onto $|0\rangle$

Let $\hat{\theta} \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\hat{\theta}|\psi\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

observable as $\hat{\theta} = \hat{\theta}^{\dagger}$

$$= \begin{pmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = |\psi\rangle$$

$$\hat{\theta}|\psi\rangle = |\psi\rangle$$

eigenvalue $= 1$

$|\psi\rangle$ is an eigenstate of $\hat{\theta}$!

$\langle\psi|\psi\rangle = 1$

Normalized

$\hat{\theta} = \hat{\theta}^\dagger$ check!

What are the eigenstates of $\hat{\theta}$?

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \lambda \hat{I} = \underline{0}$$

$$\Rightarrow \det \left| \hat{\theta} - \lambda \hat{I} \right| = 0$$

$$\Rightarrow \lambda = \pm 1$$

$$\lambda = +1 \Rightarrow \begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -x = iy \; ; -ix = y$$

$\lambda = 1$ e.v. $\Rightarrow \begin{pmatrix} 1 \\ i \end{pmatrix}$ or $\begin{pmatrix} i \\ -1 \end{pmatrix}$ or, $x = i$, $y = -1$

$\lambda = -1$ e.v. $\Rightarrow \begin{pmatrix} i \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

Linear operator

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -i \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

new bases

$|\theta_1\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} i \\ 1 \end{pmatrix} = |\theta_2\rangle$ forms a basis

$$\langle\theta_1|\theta_1\rangle = (1 \; -i)\begin{pmatrix} 1 \\ i \end{pmatrix} = 2$$

they are not normalized!

$$|\tilde{\theta}_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|\tilde{\theta}_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

→ orthonormal vectors

$$\langle\tilde{\theta}_1|\tilde{\theta}_2\rangle = \underline{0}$$

$$|\tilde{\theta}_1\rangle\langle\tilde{\theta}_1| + |\tilde{\theta}_2\rangle\langle\tilde{\theta}_2| = \hat{\mathbb{1}}$$ check!

$$|\tilde{\theta}_1\rangle\langle\tilde{\theta}_1| = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}(1 \; -i) = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\frac{1}{2}$$

$$|\tilde{\theta}_2\rangle\langle\tilde{\theta}_2| = \frac{1}{2}\begin{pmatrix} i \\ 1 \end{pmatrix}(-i \; 1) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Adding two terms yields $\hat{\mathbb{1}}$.

$$\hat{\theta} = \sum_{n=1}^{2} \lambda_n |\tilde{\theta}_n\rangle\langle\tilde{\theta}_n| = \sum_{n=1}^{2} \lambda_n P_n$$

$\downarrow$ Projector

$$|\psi\rangle = \sum_{n=1}^{2} a_n |\tilde{\theta}_n\rangle \qquad \text{check!}$$

$$\langle\tilde{\theta}_1|\psi\rangle = a_1 = \frac{1}{\sqrt{2}}(1\ \ -i)\begin{pmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{2}(1\ \ -i)\begin{pmatrix} i \\ -1 \end{pmatrix}$$

$$= i$$

$$\langle\tilde{\theta}_2|\psi\rangle = a_2 = \frac{1}{2}(-i\ \ 1)\begin{pmatrix} i \\ -1 \end{pmatrix}$$

$$= 0$$

$\therefore$ $|\psi\rangle$ is already an eigenstate of $\hat{\theta}$ so $P(a_1) = 1 = |i|^2$

Made with Goodnotes

If $\hat{O}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\hat{O}_2 |\psi\rangle = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \rightarrow \neq |\psi\rangle$$

$\hat{O}$ has two eigenvalues :- 1 and -1.
with eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|\psi\rangle = \sum_n a_n |e_n\rangle$$

This is just a number. NOT an eigenvalue!!

$$= \frac{i}{\sqrt{2}} |e_1\rangle - \frac{1}{\sqrt{2}} |e_2\rangle$$

$$= \frac{i}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$$

$$P(a_1) = |\langle 0|\psi\rangle|^2 = \frac{1}{2}$$

$$P(a_2) = |\langle 1|\psi\rangle|^2 = \frac{1}{2}$$

$|\psi\rangle$ is expressed as a linear superposition of the eigenstates of $\hat{O}_2$.

Exc: Check for $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Exc:- Show $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ is a unitary matrix, i.e. $U^{\dagger} U = \mathbb{1}$

* defined later

Also

$$\hat{O}_2 = \sum_{n=1}^{2} \lambda_n P_n$$

$$P_n = |\phi_n\rangle\langle\phi_n|$$

Matrix element of $\hat{A}$:

$$A_{ij} = \langle i | A | j \rangle$$

$|i\rangle, |j\rangle$ are basis vectors

* Linear operators

If A is an operator that maps a vector space V to W and satisfy

(i) $v \in V$ and $\alpha \in \mathbb{C}$

$$A\{\alpha|v\rangle\} = \alpha A|v\rangle$$

and $A|v\rangle \in W$

(ii) $A(\alpha_1|v_1\rangle + \alpha_2|v_2\rangle) = \alpha_1 A|v_1\rangle + \alpha_2 A|v_2\rangle$

Then A is called a linear operator

Adjoint of operator

$$\langle\psi|O_2|\phi\rangle = \langle\phi|O_2^{\dagger}|\psi\rangle^*$$

Check

$$|\psi\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$|\phi\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

L.H.S. $= \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

R.H.S. $= \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$

$= \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$

Kernel, Rank etc.

(Extra study)

Let $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\operatorname{Ker}(J) := \{ x \in V : Jx = 0 \}$$

$$J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Write in row reduced form:-

$C_1' = C_2 - C_1 \downarrow$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{C_2' = C_2 - C_3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1' = R_2 - R_1$

$R_2' = R_2 - R_1$

$\underline{\underline{J'}}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow x_3 = 0$$

ES

Null space of J

$$\mathcal{N}[J] = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

$\text{rank}(J) = 1$

$\text{Nullity}(J) = \dim \text{ of } \ker(J)$

$= 2$

$\text{rank}(J) + \text{Nullity}(J) = \dim(V)$

$1 + 2 = 3$

Trace of $\hat{A}$

$\hat{A}_{ij} = \langle i | A | j \rangle$

$$\text{Tr}\,\hat{A} = \sum_i \hat{A}_{ii}$$

Show:- $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$(AB)_{ij} = \langle i | AB | j \rangle$$

$$\mathrm{Tr}\, AB = \sum_i \langle i | AB | i \rangle$$

use completeness relation

$$\sum_{i,j} \langle i | A | j \rangle \langle j | B | i \rangle$$

$$= \sum_{i,j} \langle j | B | i \rangle \langle i | A | j \rangle$$

$$= \sum_j \langle j | BA | j \rangle = \mathrm{Tr}\, BA$$

Unitary operator:- $U^\dagger U = UU^\dagger = \mathbb{1}$

transformations that preserves the norm of states/bases

$|i\rangle \to |i'\rangle = U|i\rangle \Rightarrow \langle i'|i'\rangle = \langle i | U^\dagger U | i \rangle = \langle i | i \rangle$

Eigenvalues of a Hermitian operator are real.

Proof:- Let $\hat{A}$ be a Hermitian operator and $|a_i\rangle$ $i = 1, \dots, n$ are its eigenkets (normalized)

$$\hat{A}\,|a_i\rangle = a_i\,|a_i\rangle \quad (1)$$

also, $$\hat{A}\,|a_j\rangle = a_j\,|a_j\rangle \quad (2)$$

$$(2) \Rightarrow \langle a_j|\,\hat{A}^\dagger = a_j^*\,\langle a_j|$$
$$= \langle a_j|\,\hat{A}$$

$$\therefore \quad \langle a_j|\,\hat{A}\,|a_i\rangle = a_j^*\,\langle a_j|a_i\rangle$$

and $$\langle a_j|\,\hat{A}\,|a_i\rangle = a_i\,\langle a_j|a_i\rangle$$

taking difference and setting $i = j$ we get $a_i = a_i^*$

$$\therefore \quad (a_i - a_j^*)\,\langle a_j|a_i\rangle = 0$$ and $\langle a_i|a_i\rangle = 1$

Made with Goodnotes

Proof:- eigenstates of distinct eigenvalues are orthogonal

$$(a_i - a_j^*)\langle a_j | a_i\rangle = 0$$

for $i \neq j$

we get $a_i \neq a_j$

so $\langle a_j | a_i \rangle = 0$

* All observables are Hermitian operators.

Unitary operator explicit construction :-

Let, $U|e^{(k)}\rangle = |f^{(k)}\rangle \rightarrow$ another basis $k=1,\dots,n$

→ braket

$$U = \sum_{k=1}^{n} |e^{(k)}\rangle\langle f^{(k)}| \text{ is unitary}$$

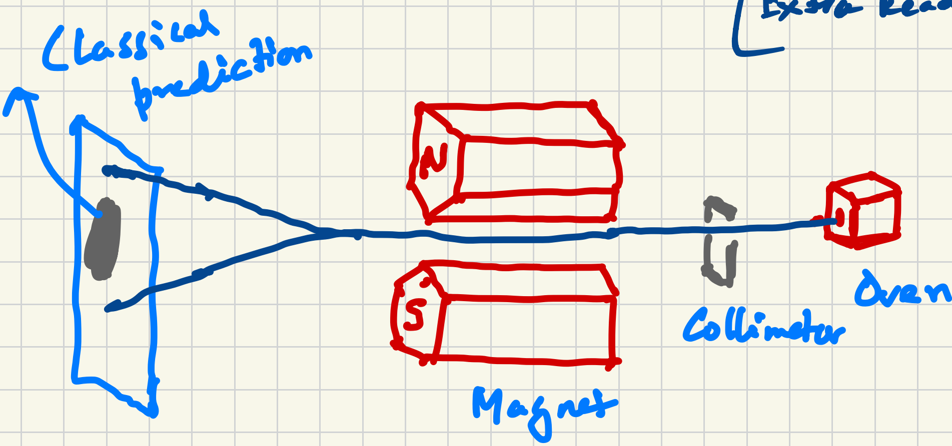
$$U^\dagger = \sum_k |f^{(k)}\rangle\langle e^{(k)}|$$

$$U^\dagger U = \sum_{k,l} |f^{(k)}\rangle \underbrace{\langle e^{(k)}| e^{(l)}\rangle}_{\delta_{kl}} \langle f^{(l)}|$$

$$= \sum_k |f^{(k)}\rangle\langle f^{(k)}| = \hat{1}$$

Stern-Gerlach Experiment

(Extra Reading)



Ag atoms heated in an oven

Inhomogeneous magnetic field is produced through magnetic pole pieces. The 5S e^- of Ag has a magnetic moment. Ag atom possesses a net Mag. moment

$$\vec{\mu} = \frac{e}{m_e c}\vec{S} \qquad \vec{\mu} \propto \vec{S}$$

$\vec{S}$ → e^- spin ang momentum

If the mag. field $\vec{B}$ is in z direction

$$F_z = \mu_z \frac{\partial B_z}{\partial z}$$

ER

Unlike classical particle e^-/Ag atoms coming out of the SG apparatus splits into two parts.

S_z up and down

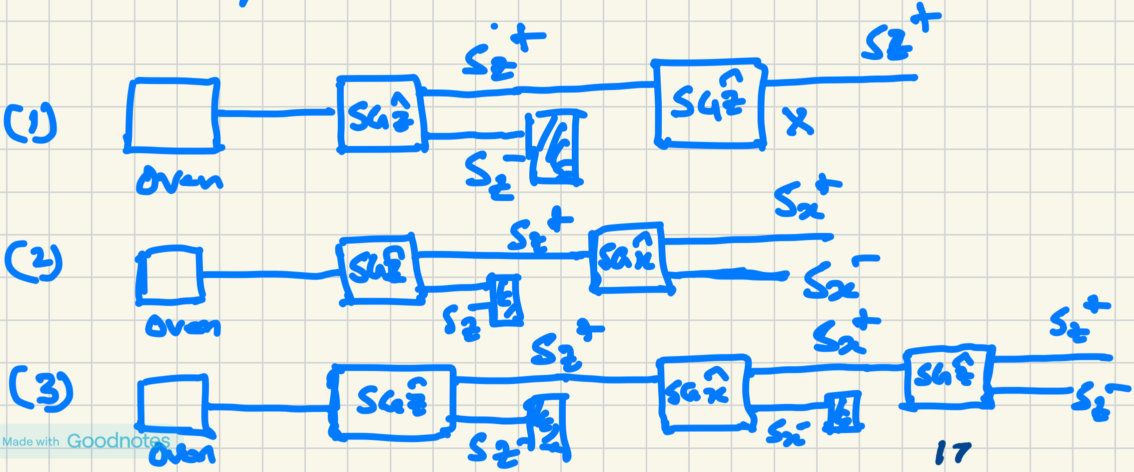
S_z+ and S_z-

Classically all values of μ_z between $|\mu|$ and $-|\mu|$ could have been possible.

Observed values of S_z are numerically equal to $\pm \frac{\hbar}{2} \longrightarrow$ Planck constant

'Space Quantization'

e^- spin is quantized.



Analogy with polarization of light

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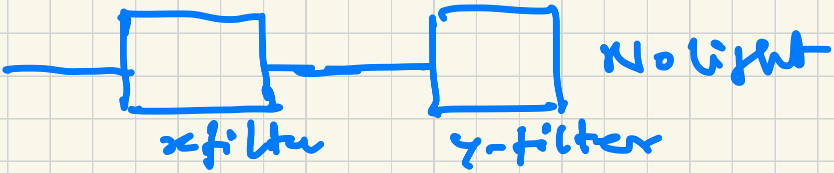
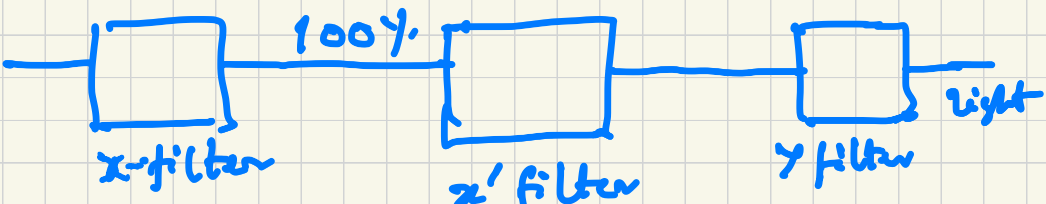


$\vec{E} = E_0 \hat{x} \cos(kz - \omega t)$

Electric field.

$\vec{E} = E_0 \hat{y} \cos(kz - \omega t)$



$$E_0 \hat{x}' \cos(kz - \omega t) = E_0 \left[\frac{\hat{x}}{\sqrt{2}} \cos(\ ) + \frac{\hat{y}}{\sqrt{2}} \cos(\ ) \right]$$

$(\ ) \equiv kz - \omega t$

$$E_0 \hat{y}' \cos(kz - \omega t) = E_0 \left[-\frac{\hat{x}}{\sqrt{2}} \cos(\ ) + \frac{\hat{y}}{\sqrt{2}} \cos(\ ) \right]$$

ER

So

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}|S_z;+\rangle + \frac{1}{\sqrt{2}}|S_z;-\rangle$$

$$|S_x;-\rangle = -\frac{1}{\sqrt{2}}|S_z;+\rangle + \frac{1}{\sqrt{2}}|S_z;-\rangle$$

For $|S_y;\pm\rangle$ states we dont have any option with real co-efficients!! otherwise they are identical!

$|S_y;\pm\rangle$ states can be constructed in analogy with a circularly polarized light where

$$\vec{E} = E_0\left[\frac{1}{\sqrt{2}}\hat{x}\cos(\;) + \frac{1}{\sqrt{2}}\hat{y}\cos(kz-\omega t + \pi/2)\right]$$

$$e^{i\pi/2}$$

$$Re(\vec{E}) = \frac{\vec{E}}{E_0}$$

$$\vec{E} = \frac{\hat{x}}{\sqrt{2}}e^{i(kz-\omega t)} + \frac{\hat{y}\,i}{\sqrt{2}}e^{i(kz-\omega t)}$$

So $|S_y;\pm\rangle = \frac{1}{\sqrt{2}}|S_z;+\rangle \pm \frac{i}{\sqrt{2}}|S_z;-\rangle$

Uncertainty Relation :-

If $\hat{A}$ is Hermitian, it has real eigenvalues

For an anti-Hermitian operator $\hat{B} = -\hat{B}^\dagger$; the eigenvalues are purely imaginary.

Proof:- Let $|b_i\rangle$s are (normalized) eigenkets of $\hat{B}$

$$\hat{B}|b_i\rangle = b_i|b_i\rangle \quad (1)$$

$$(1) \Rightarrow \langle b_i|\hat{B}^\dagger = \langle b_i|b_i^*$$

$$\langle b_i|\hat{B}^\dagger|b_i\rangle = \langle b_i|b_i^*|b_i\rangle \quad (3)$$

$$(1) \Rightarrow \langle b_i|\hat{B}|b_i\rangle = b_i\langle b_i|b_i\rangle \quad (4)$$

$$\because \hat{B} = -\hat{B}^\dagger \quad (3)+(4) \Rightarrow 0 = (b_i^* + b_i)$$

$$\Rightarrow \boxed{b_i = -b_i^*}$$

$\Rightarrow$ b_i is purely imaginary.

Expectation of an Operator $\hat{A}$:-

$$\langle \hat{A} \rangle = \langle \alpha | \hat{A} | \alpha \rangle$$

↳ state ket $|\alpha\rangle$

The system is described by $|\alpha\rangle$.

$$\langle \hat{A} \rangle = \langle \alpha | \hat{A} \Big[\sum_{i=1}^{n} \underbrace{|a_i\rangle}_{\text{eigenket of } \hat{A}} \langle a_i | \alpha \rangle \Big]$$

$$= \sum_{i=1}^{n} a_i \langle \alpha | a_i \rangle$$

$$= \sum_i a_i \, |\langle a_i | \alpha \rangle|^2$$

$$= \sum_i a_i \; P(a_i) \rightarrow \text{probability of finding } \hat{A} \text{ in the state } |a_i\rangle.$$

↳ eigenvalue or measured value

* Ideally the expectation of an operator is obtained by preparing 'n' identical copies of the system.

Ex:-

$$|\psi\rangle \xrightarrow{\text{measurement}} |a_i\rangle \text{ if}$$

one measures the observable $\hat{A}$.

(Extra Reading): Pg 22-28

A beam of silver atom coming out of the oven having an arbitrary spin orientation w.r.t. a SG $\hat{z}$ aparatus may change into either $|S_z;+\rangle$ or $|S_z;-\rangle$ state.

We often denote these states as $|+\rangle$ and $|-\rangle$

$$\hat{S}_z \doteq \frac{\hbar}{2}\left[|+\rangle\langle+| - |-\rangle\langle-|\right]$$

Represent $|+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}$ and $|-\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$

Show $|+\rangle\langle+| + |-\rangle\langle-| = \mathbb{1}$

$\left[\text{using } \hat{S}_z = \sum_i a_i |a_i\rangle\langle a_i|\right]$ Spectral decomposition

and $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$

$$\hat{S}_z \equiv \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$|S_x;\pm\rangle \doteq \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$$

$$|S_y;\pm\rangle \doteq \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$$

Find $\hat{S}_x$ and $\hat{S}_y$

Expectation of $\hat{S}_z$ in state $\cos\theta|+\rangle + \sin\theta|-\rangle$

i.e. $$\langle \hat{S}_z \rangle = \frac{\hbar}{2}|\langle + | \cos\theta | + \rangle|^2 - \frac{\hbar}{2}|\langle - | \sin\theta | - \rangle|^2$$

$$= \frac{\hbar}{2}\cos 2\theta$$

Schwarz inequality :- $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

Proof :- take $\| |\alpha\rangle + \lambda|\beta\rangle \|^2 \geq 0$

$$\Rightarrow \left(\langle\alpha| + \lambda^*\langle\beta|\right)\left(|\alpha\rangle + \lambda|\beta\rangle\right) \geq 0$$

choose $\lambda = -\dfrac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$

$$\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$$

dispersion of $\hat{A}$ $\langle(\Delta\hat{A})^2\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$

Commutator of operator

$$XY \neq YX$$

For Matrices or Operators.

$$XY - YX \equiv [X, Y]$$

Let, $\Delta\hat{A}|\ \rangle = |\alpha\rangle$

any ket

$$\Delta\hat{B}|\ \rangle = |\beta\rangle$$

then C.S. inequality gives

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq |\langle\Delta\hat{A}\Delta\hat{B}\rangle|^2$$

Now,

$$\Delta\hat{A}\Delta\hat{B} = \frac{1}{2}[\Delta\hat{A}, \Delta\hat{B}] + \frac{1}{2}\underbrace{\{\Delta\hat{A}, \Delta\hat{B}\}}_{\substack{\text{anti-commutator} \\ AB+BA}}$$

$$[\Delta\hat{A}, \Delta\hat{B}] = [\hat{A} - \langle\hat{A}\rangle, \hat{B} - \langle\hat{B}\rangle]$$

$$= [\hat{A}, \hat{B}]$$

$$[\hat{A}, \hat{B}]^{\dagger} = (\hat{A}\hat{B} - \hat{B}\hat{A})^{\dagger}$$

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger}$$

$$= \hat{B}^{\dagger}\hat{A}^{\dagger} - \hat{A}^{\dagger}\hat{B}^{\dagger}$$

$$= \hat{B}\hat{A} - \hat{A}\hat{B}$$

$$= -[\hat{A}, \hat{B}] \rightarrow \text{Anti Hermitian}$$

$$\{\Delta\hat{A}, \Delta\hat{B}\} = \{\hat{A} - \langle\hat{A}\rangle, \hat{B} - \langle\hat{B}\rangle\}$$

$$= \{\hat{A}, \hat{B}\} \quad \text{Hermitian}$$

$$- 2\hat{A}\langle\hat{B}\rangle - 2\hat{B}\langle\hat{A}\rangle$$

$$\therefore \langle\Delta\hat{A}\Delta\hat{B}\rangle = \frac{1}{2}\langle[\hat{A}, \hat{B}]\rangle + \frac{1}{2}\langle\{\Delta\hat{A}, \Delta\hat{B}\}\rangle$$

$\therefore \quad |\langle \Delta\hat{A}\Delta\hat{B}\rangle|^2 = \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2$
$+ \frac{1}{4}|\langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle|^2$

$\therefore$ C.S. $\Rightarrow$

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2$$

Check

$$\hat{S}_x \doteq \hat{S}_x\left(|S_x;+\rangle\langle S_x;+| + |S_x;-\rangle\langle S_x;-|\right)$$

$$\doteq \frac{\hbar}{2}\left[|S_x;+\rangle\langle S_x;+| - \frac{\hbar}{2}|S_x;-\rangle\langle S_x;-|\right]$$

$$\doteq \frac{\hbar}{2}\left[\frac{1}{2}\left(|+\rangle\langle+| + |+\rangle\langle-| + |-\rangle\langle-| + |-\rangle\langle-|\right)\right.$$
$$\left. - \frac{1}{2}\left(|+\rangle\langle+| - |+\rangle\langle-| - |-\rangle\langle+| + |-\rangle\langle-|\right)\right]$$

$$= \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)$$

$\langle \hat{S}_x \rangle$ in $|+\rangle$ state

$$\langle \hat{S}_x \rangle = \frac{\hbar}{2} \langle + | \hat{S}_x | + \rangle$$

$$= \frac{\hbar}{2} \times 0 = 0$$

$$\langle \hat{S}_x^2 \rangle = \frac{\hbar^2}{4}$$

$$\because \quad \hat{S}_x |+\rangle = \frac{\hbar}{2} |-\rangle$$

$$\langle + | \hat{S}_x^{\dagger} = \langle - | \frac{\hbar}{2}$$

$$\langle + | \hat{S}_x^{\dagger} \hat{S}_x | + \rangle = \frac{\hbar^2}{4}$$

$$\therefore \langle (\hat{S}_x - \langle \hat{S}_x \rangle)^2 \rangle = \langle \Delta \hat{S}_x^2 \rangle = \frac{\hbar^2}{4}$$

Eg:

$$[x, p_x] = i\hbar$$

$$\langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \geq \frac{|\langle[x, p_x]\rangle|^2}{4}$$

$$\langle\Delta x^2\rangle\langle\Delta p^2\rangle \geq \frac{\hbar^2}{4}$$

$$\Rightarrow \quad \Delta x\, \Delta p \geq \frac{\hbar}{2}$$

Pag

Completeness Relation:-

$$|\psi\rangle = \sum_{k=1}^{n} c_k |a_k\rangle$$

→ complex co-efficient

"from Quantum postulate"

Superposition of eigenstates of an operator.

$$|\psi\rangle = \sum_k \langle a_k|\psi\rangle |a_k\rangle$$

$$= \sum_k \underbrace{|a_k\rangle\langle a_k|}_{\hat{\mathbb{1}}}\psi\rangle$$

$$\Rightarrow \boxed{\sum_k |a_k\rangle\langle a_k| = \hat{\mathbb{1}}}$$

Spectral decomposition :-

∵ Any Hermitian operator $\hat{A}$ has an orthonormal set of eigenkets. These kets form a basis.

$$\hat{A}.\hat{\mathbb{1}} = \sum_k \hat{A}|a_k\rangle\langle a_k|$$

$$\Rightarrow \boxed{\hat{A} = \sum_k a_k |a_k\rangle\langle a_k|}$$

→ spectral decomposition

Projection Operator example

① $|e_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$; $|e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

express $|\psi\rangle = \begin{pmatrix}3\\5\end{pmatrix}$ in terms of P_1 and P_2.

② Spectral Theorem

Show that if $\hat{A} = \sum_i \lambda_i P_i$ (projector)

then $\hat{A}^n = \sum_i \lambda_i^n P_i^n$

If A is an operator that maps $(|e_1\rangle, |e_2\rangle, \ldots |e_n\rangle)$ basis vectors to $(|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle)$ vectors. Then A's matrix representation is :-

$$A \doteq \sum_{i=1}^{n} |f_i\rangle\langle e_i|$$

③ Find the matrix A that maps $|0\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|1\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.