Numerical Methods

IIIT, Allahabad

Instructor: Srijit Bhattacharjee

Linear Algebra Primer

- Linear transformations on a Vector space can be represented by matrices.
- If T is the linear transformation on V, a vector space of dimension n, and if V has a basis $\mathcal{B} = \{b_i\}$, then since every vector in V can be written as linear combination of others we have

$$Tb_j = \sum_{i=1}^{n} t_{ij} b_i.$$

  t_{ij}s are n^2 scalars. They form a matrix T.
- Matrix Inverse has applications in solving simultaneous equations
- A non-singular matrix M has its inverse M^{-1} satisfying:

$$M^{-1}M = I = MM^{-1}$$

  and $M_{ij}^{-1} = \frac{1}{detM} C_{ij}^{T}$. where C is the cofactor matrix.

Gauss-Siedel Method

- Consider the system of linear equations:

$$2x + z = 9$$
$$2x + y - z = 6$$
$$3x + y - z = 9$$

- In matrix form $AX = C$. Solution can be obtained by inverting the matrix $\mathrm{X} = A^{-1}C$. show the unique solution is $x = 3$, $y = 3$, $z = 3$.
- Alternatively one can employ **Gauss-Siedel method**
  1. First start with a trial solution $\mathrm{X}^T = (-, 0, 0)$.
  2. Iterate by rearranging the rows in the following way:

$$x = -\frac{1}{2}z + \frac{9}{2}$$
$$y = -2x + z + 6$$
$$z = 3x + y - 9$$

- Re-arrange the equations in diagonally dominant form

GS method contd.

- Subtract first and second eqn. $2z - y = 3$. This now becomes a new equation $z = 1/2y + 3/2$
- Now subtracting last and second equation $\implies x = 3$ and then putting this into second equation we get $y - z = 0$.
- Equations in diagonally dominant form:

$$x = -\frac{1}{2}z + \frac{9}{2}$$
$$y = z$$
$$z = \frac{1}{2}y + \frac{3}{2}$$

- After two iterations we get approximate solution:$x = 2.875, y = 2.625, z = 2.625$.
- Make use of a tolerance: $|x_i^{(k+1)} - x_i^{(k)}| < \epsilon$, to iterate till one gets desired accuracy. For example $\epsilon \sim 10^{-2}$.

GS method contd.

- The iteration may not converge if the equations are not in the diagonally dominant form. Check!
- In diagonally dominant form the GS method algorithm reads:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right],$$

$i = 1, 2, \cdots, n; \; k = 0, 1, 2, \cdots$

GS method in matrix form

- The GS method is described by the following matrix form:

$$X^{(k+1)} = L^{-1}(b - UX^{(k)})$$

- Where we used $A = L + U$, U is a strictly upper triangular matrix.
- Compared to Gauss elimination method Gauss-Siedel method converges more rapidly in most of the times.

Eigenvalue and Eigenvector

- A non-null vector X of a matrix linear operator A is called an eigenvector if the matrix satisfies:

  $$AX = aX.$$

  a is called the eigenvealue corresponding to the eigenvector X.
- If A has a set of n eigenvectors X_i with eigenvalues a_is$(1 = 1, 2, \cdots, n)$, then $B =: (X_1|X_2|\cdots|X_n)$ diagonalizes A with:

  $$A_d = B^{-1}AB$$

- If a is an eigenvalue of A with eigenvector X then a^3 is eigenvalue of A^3 with the same eigenvector.