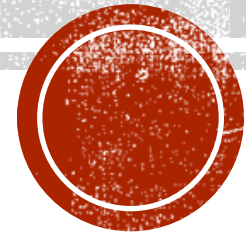




Indian Institute of Information Technology Allahabad

Data Structures and Algorithms

Heap Sort and Priority Queue



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Priority Queue ADT

- ◆ A priority queue stores a collection of entries
- ◆ Each **entry** is a pair (key, value)
- ◆ Main methods of the Priority Queue ADT
 - **insert**(k, x)
inserts an entry with key k and value x
 - **removeMin**()
removes and returns the entry with smallest key

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inserts an entry with key k and value x
 - **removeMin()**
removes and returns the entry with smallest key
- ◆ Additional methods
 - **min()**
returns, but does not remove, an entry with smallest key
 - **size(), isEmpty()**
- ◆ Applications:
 - Standby flyers
 - Auctions
 - Stock market

Implementing Priority Queue with Linked Lists

◆ Implementation with an unsorted list



◆ Performance:

- **insert** takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
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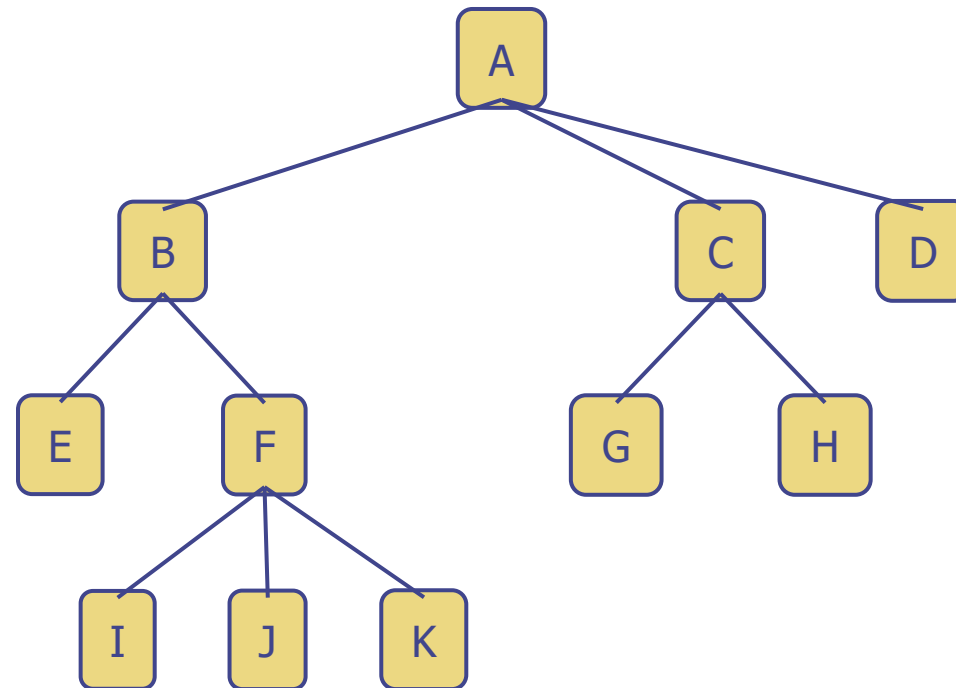


- ◆ Performance:

- **insert** takes $O(n)$ time since we have to find the place where to insert the item
- **removeMin** and **min** take $O(1)$ time, since the smallest key is at the beginning

Can we do better?

Yes, using *Heaps*,
which are built using *Trees* :



Heap Data Structure

- Array can be viewed as a nearly complete binary tree.
 - **Physically** – linear array.
 - **Logically** – binary tree, filled on all levels (except lowest.)
- **Map from array elements to tree nodes and vice versa**
 - Root – $A[1]$
 - Left[i] – $A[2i]$
 - Right[i] – $A[2i+1]$
 - Parent[i] – $A[\lfloor i/2 \rfloor]$

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 - Left[i] – $A[2i]$
 - Right[i] – $A[2i+1]$
 - Parent[i] – $A[\lfloor i/2 \rfloor]$
- **length[A]** – number of elements in array A .
- **heap-size[A]** – number of elements in heap stored in A .
 - **heap-size[A] \leq length[A]**

Heap Property (Max and Min)

- Max-Heap

- For every node excluding the root, value is at most that of its parent: $A[\text{parent}[i]] \geq A[i]$

- Largest element is stored at the root.

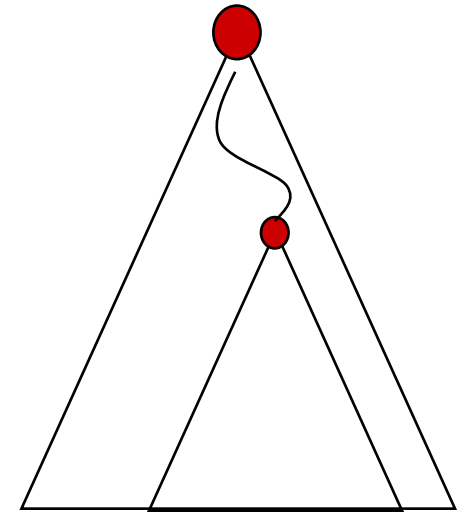
- In any subtree, no values are larger than the value stored at subtree root.

- Min-Heap

- For every node excluding the root, value is at least that of its parent: $A[\text{parent}[i]] \leq A[i]$

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- In any subtree, no values are smaller than the value stored at subtree root

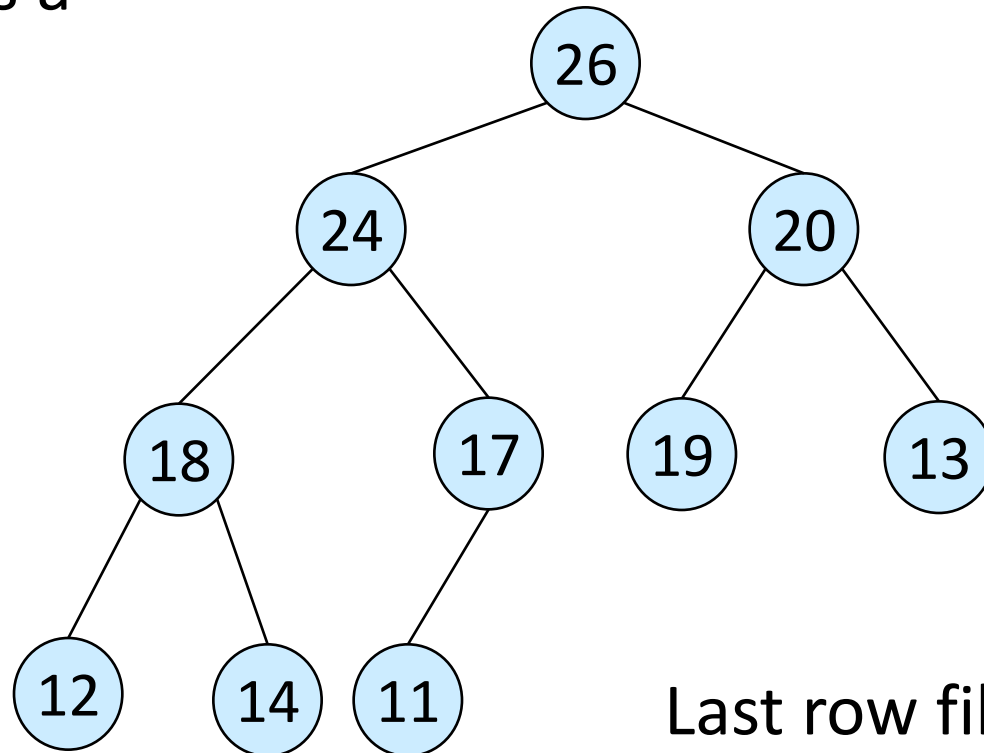


Heaps – Example

26	24	20	18	17	19	13	12	14	11
1	2	3	4	5	6	7	8	9	10

Max-heap as an array.

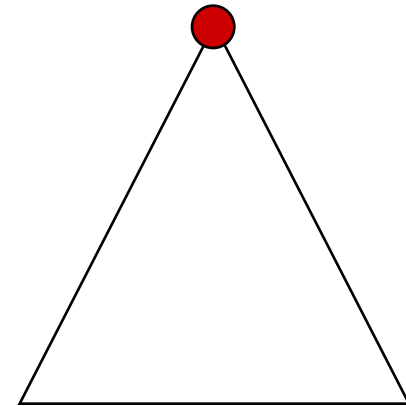
Max-heap as a binary tree.



Last row filled from left to right.

Height

- *Height of a node in a tree*: the number of edges on the longest simple downward path from the node to a leaf.
- *Height of a tree*: the height of the root.
- *Height of a heap*: $\lfloor \log n \rfloor$
 - Basic operations on a heap run in $O(\log n)$ time



Heapsort

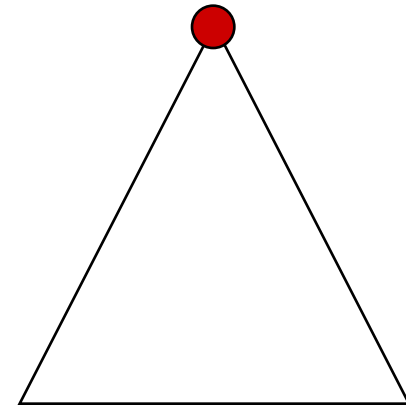
- Combines the better attributes of merge sort and insertion sort.
 - Like merge sort, but unlike insertion sort, running time is $O(n \lg n)$.
 - Like insertion sort, but unlike merge sort, sorts in place.
- Introduces an algorithm design technique
 - Create data structure (*heap*) to manage information during the execution of an algorithm.
- The *heap* has other applications beside sorting.
 - Priority Queues

Heaps in Sorting

- Use **max-heaps for sorting**.
- The array representation of max-heap is not sorted.
- **Steps in sorting**
 - Convert the given array of size n to a max-heap (*BuildMaxHeap*)
 - Swap the first and last elements of the array.
 - Now, the largest element is in the last position – where it belongs.
 - That leaves $n - 1$ elements to be placed in their appropriate locations.
 - However, the array of first $n - 1$ elements is no longer a max-heap.
 - Float the element at the root down one of its subtrees so that the array remains a max-heap (*MaxHeapify*)
 - Repeat step 2 until the array is sorted.

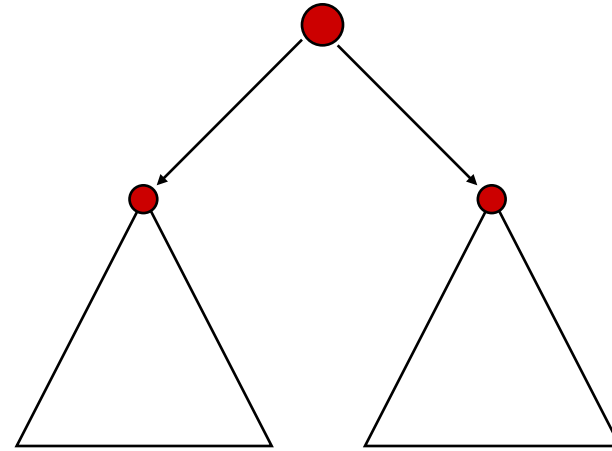
Heap Characteristics

- *Height* = $\lfloor \log n \rfloor$ i.e., floor(log n)
- No. of *leaves* = $\lceil n/2 \rceil$ i.e., ceil(log n)
- No. of nodes of height h $\leq \lceil n/2^{h+1} \rceil$



Maintaining the heap property

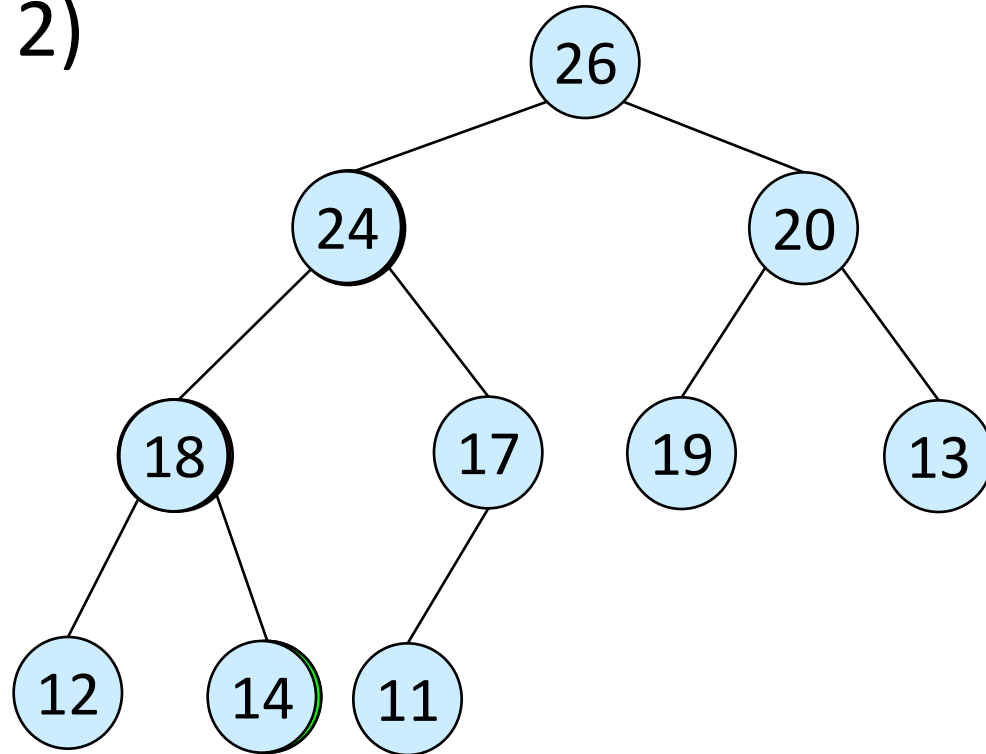
- Suppose two subtrees are max-heaps, but the root violates the max-heap property.



- **Fix** the offending node by exchanging the value at the node with the larger of the values at its children.
 - May lead to the subtree at the child not being a heap.
- **Recursively fix the children** until all of them satisfy the max-heap property.

MaxHeapify – Example

MaxHeapify(A, 2)



Procedure MaxHeapify

MaxHeapify(A, i)

1. $l = \text{left}(i)$
2. $r = \text{right}(i)$
3. **if** $l \leq \text{heap-size}[A]$ and $A[l] > A[i]$
4. **then** $\text{largest} = l$
5. **else** $\text{largest} = i$
6. **if** $r \leq \text{heap-size}[A]$ **and** $A[r] > A[\text{largest}]$
7. **then** $\text{largest} = r$
8. **if** $\text{largest} \neq i$
9. **then** exchange $A[i] \leftrightarrow A[\text{largest}]$
10. $\text{MaxHeapify}(A, \text{largest})$

Assumption:

$\text{Left}(i)$ and $\text{Right}(i)$
are max-heaps.

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Time to fix node i and
its children = $\Theta(1)$

PLUS

Time to fix the subtree
rooted at one of i 's
children = $T(\text{size of}$
subtree at $\text{largest})$

Running Time for MaxHeapify(A, n)

- MaxHeapify takes $O(h)$ where h is the height of the node where MaxHeapify is applied
- Alternately, $T(n) = O(\log n)$ in worst case

Building a heap

- Use *MaxHeapify* to convert an array A into a max-heap.
- How?

Building a heap

- Use *MaxHeapify* to convert an array A into a max-heap.
- How?
- Call *MaxHeapify* on each element in a bottom-up manner.

BuildMaxHeap(A)

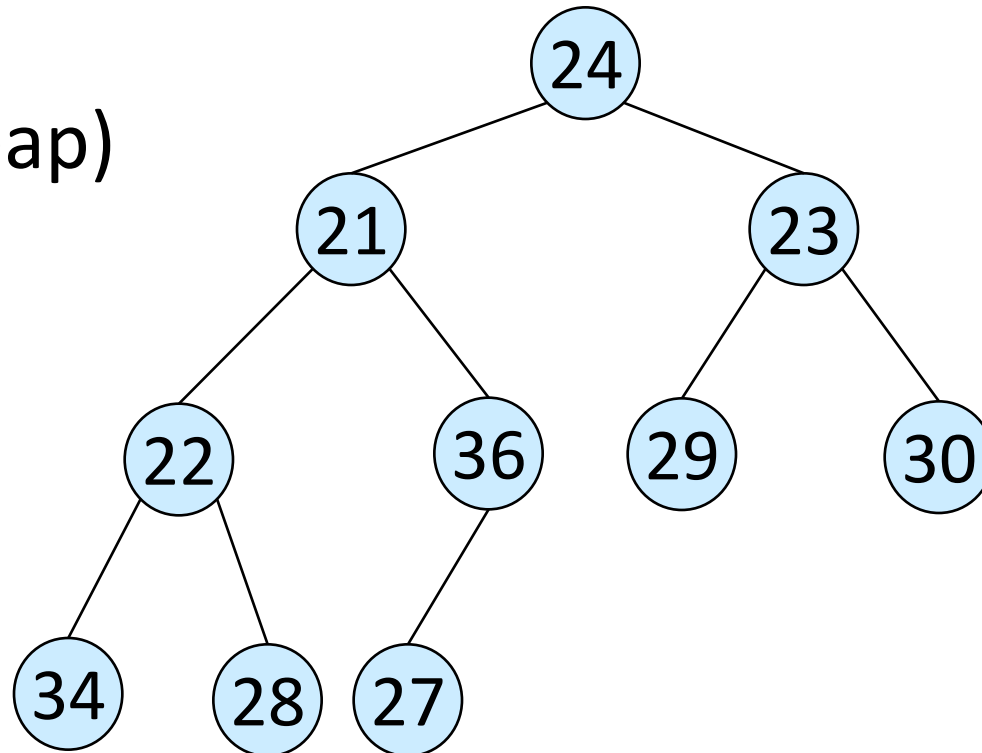
1. $heap\text{-}size[A] = length[A]$
2. **for** $i = \lfloor length[A]/2 \rfloor$ **downto** 1
3. **do** *MaxHeapify*(A, i)

BuildMaxHeap – Example

Input Array:

24	21	23	22	36	29	30	34	28	27
----	----	----	----	----	----	----	----	----	----

Initial Heap:
(not max-heap)



BuildMaxHeap – Example

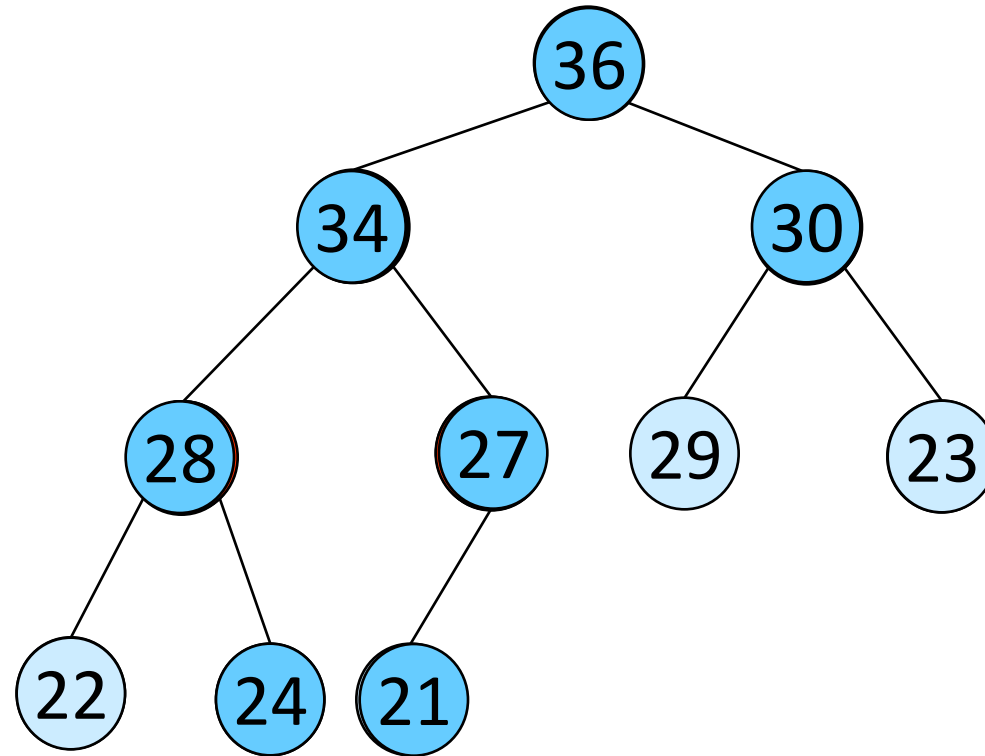
MaxHeapify($\lfloor 10/2 \rfloor = 5$)

MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

MaxHeapify(1)



Correctness of *BuildMaxHeap*

- Loop Invariant: At the start of each iteration of the **for** loop, each node $i+1, i+2, \dots, n$ is the root of a max-heap.

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Correctness of *BuildMaxHeap*

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- Initialization:
 - Before first iteration $i = \lfloor n/2 \rfloor$
 - Nodes $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \dots, n$ are leaves and hence roots of max-heaps.
- Maintenance:
 - By Loop Invariant, subtrees at children of node i are max heaps.
 - Hence, $\text{MaxHeapify}(i)$ renders node i a max heap root (while preserving the max heap root property of higher-numbered nodes).
 - Decrementing i reestablishes the loop invariant for the next iteration.

Running Time of *BuildMaxHeap*

- Loose upper bound:
 - Cost of a *MaxHeapify* call \times No. of calls to *MaxHeapify*
 - $O(\log n) \times O(n) = O(n \log n)$
- Tighter bound:
 - Cost of a call to *MaxHeapify* at a node depends on the height, h , of the node – $O(h)$.
 - Height of most nodes smaller than n .
 - Height of nodes h ranges from 0 to $\lfloor \log n \rfloor$.
 - No. of nodes of height h is $\lceil n/2^{h+1} \rceil$

Running Time of *BuildMaxHeap*

Tighter Bound for $T(\text{BuildMaxHeap})$

$T(\text{BuildMaxHeap})$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$
$$= O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

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$$\leq \sum_{h=0}^{\infty} \frac{h}{2^h}$$
$$= \frac{1/2}{(1-1/2)^2}$$
$$= 2$$

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Can build a heap from an unordered array in linear time

Heapsort

- Sort by maintaining as yet unsorted elements as a max-heap.
- Start by building a max-heap on all elements in A .
 - Maximum element is in the root, $A[1]$.
- Move the maximum element to its correct final position.
 - Exchange $A[1]$ with $A[n]$.
- Discard $A[n]$ – it is now sorted.
 - Decrement heap-size[A].
- Restore the max-heap property on $A[1..n-1]$.
 - Call *MaxHeapify*($A, 1$).
- Repeat until heap-size[A] is reduced to 2.

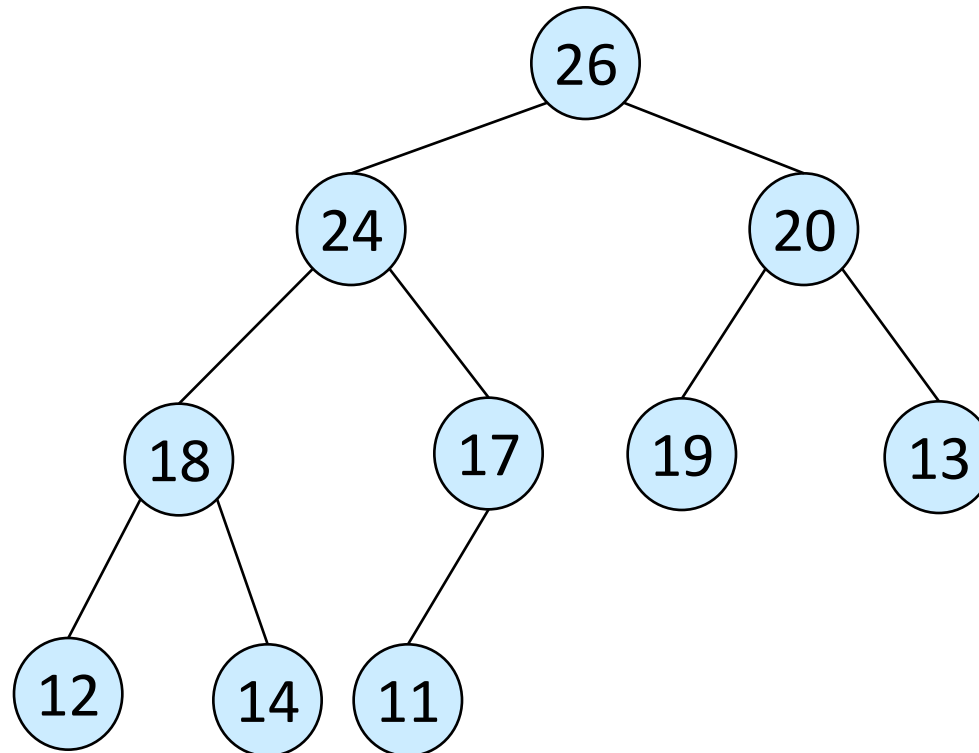
Heapsort(A)

HeapSort(A)

1. Build-Max-Heap(A)
2. **for** $i = \text{length}[A]$ **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. $\text{heap-size}[A] = \text{heap-size}[A] - 1$
5. $\text{MaxHeapify}(A, 1)$

Heapsort – Example

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Algorithm Analysis

- In-place
- Not Stable
- Build-Max-Heap takes $O(n)$ and each of the $n-1$ calls to Max-Heapify takes time $O(\log n)$.
- Therefore, $T(n) = O(n \log n)$

HeapSort(A)

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Heap Procedures for Sorting

- MaxHeapify $O(\log n)$
- BuildMaxHeap $O(n)$
- HeapSort $O(n \log n)$

Priority Queue

- Popular & important **application of heaps**.
- Max and min priority queues.
- Maintains a *dynamic* set S of elements.
- Each set element has a *key* – an associated value.
- Goal is to **support insertion and extraction efficiently**.

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- Maintains a *dynamic* set S of elements.
- Each set element has a *key* – an associated value.
- Goal is to **support insertion and extraction efficiently**.
- **Applications:**
 - Ready list of processes in operating systems by their priorities – the list is highly dynamic
 - In event-driven simulators to maintain the list of events to be simulated in order of their time of occurrence.

Basic Operations

- Operations on a max-priority queue:
 - **Insert(S, x)** - inserts the element x into the set S
 - $S \leftarrow S \cup \{x\}$.
 - **Maximum(S)** - returns the element of S with the largest key.
 - **Extract-Max(S)** - removes and returns the element of S with the largest key.
 - **Increase-Key(S, x, k)** – increases the value of element x 's key to the new value k .

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 - **Increase-Key(S, x, k)** – increases the value of element x 's key to the new value k .
- **Min-priority queue** supports **Insert**, **Minimum**, **Extract-Min**, and **Decrease-Key**.
- Heap gives a good compromise between fast insertion but slow extraction and vice versa.

Heap Property (Max and Min)

- **Max-Heap**

- For every node excluding the root, value is **at most** that of its parent: $A[\text{parent}[i]] \geq A[i]$

- **Largest** element is **stored at the root**.

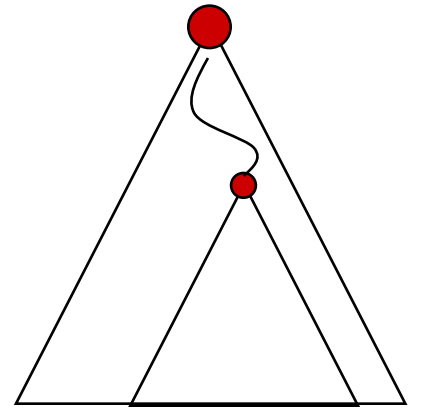
- In any subtree, no values are **larger** than the value stored at subtree root.

- **Min-Heap**

- For every node excluding the root, value is **at least** that of its parent: $A[\text{parent}[i]] \leq A[i]$

- **Smallest** element is **stored at the root**.

- In any subtree, no values are **smaller** than the value stored at subtree root



Heap-Extract-Max(A)

Implements the Extract-Max operation.

Heap-Extract-Max(A)

1. if $heap-size[A] < 1$
2. then error “heap underflow”
3. $max = A[1]$
4. $A[1] = A[heap-size[A]]$
5. $heap-size[A] = heap-size[A] - 1$
6. MaxHeapify(A, 1)
7. return max

Running time :
Dominated by the
running time of
MaxHeapify
 $= O(\log n)$

Heap-Insert(A , key)

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1. $heap-size[A] = heap-size[A] + 1$
2. $i = heap-size[A]$
3. **while** $i > 1$ **and** $A[Parent(i)] < key$
4. **do** $A[i] = A[Parent(i)]$
5. $i = Parent(i)$
6. $A[i] = key$

Running time is $O(\log n)$

The path traced from the new leaf to the root has length $O(\log n)$

Heap-Increase-Key(A, i, key)

Heap-Increase-Key(A, i, key)

```
1  If  $key < A[i]$ 
2      then error “new key is smaller than the current key”
3   $A[i] = key$ 
4  while  $i > 1$  and  $A[\text{Parent}[i]] < A[i]$ 
5      do exchange  $A[i] \leftrightarrow A[\text{Parent}[i]]$ 
6       $i = \text{Parent}[i]$ 
```

Heap-Insert(A, key)

```
1   $heap\text{-}size[A] = heap\text{-}size[A] + 1$ 
2   $A[heap\text{-}size[A]] = -\infty$ 
3   $Heap\text{-}Increase\text{-}Key(A, heap\text{-}size[A], key)$ 
```

Examples

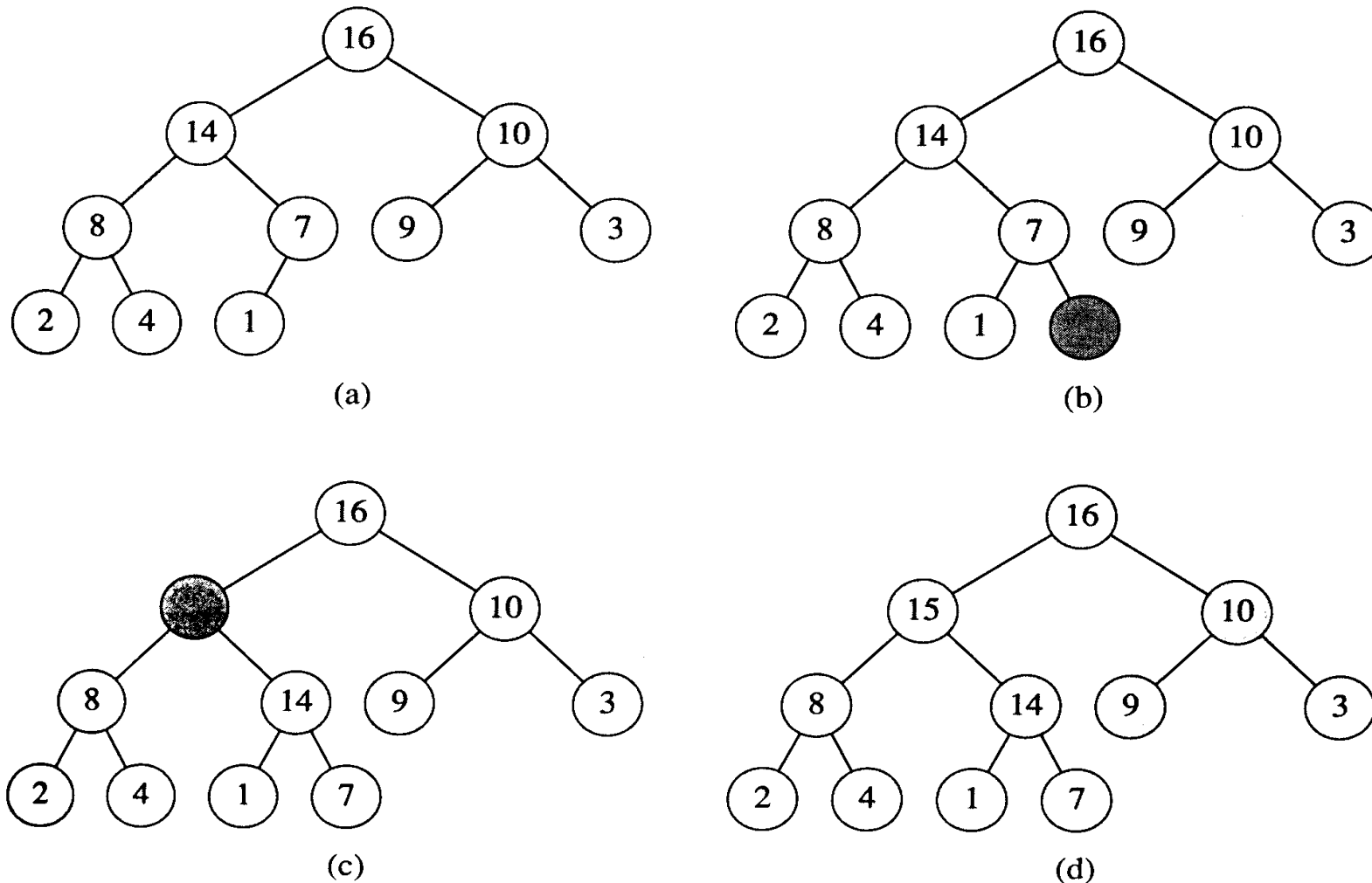


Figure 7.5 The operation of HEAP-INSERT. (a) The heap of Figure 7.4(a) before we insert a node with key 15. (b) A new leaf is added to the tree. (c) Values on the path from the new leaf to the root are copied down until a place for the key 15 is found. (d) The key 15 is inserted.

Acknowledgement

- University of North Carolina at Chapel Hill

Thank You