Indian Institute of Information Technology Allahabad

Data Structures and Algorithms

Asymptotic Analysis

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The plan

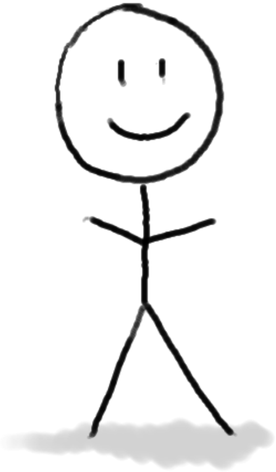
- Sorting Algorithms
  - InsertionSort: does it work and is it fast?
  - MergeSort: does it work and is it fast?
  - Skills:
    - Analyzing correctness of iterative and recursive algorithms.
    - Analyzing running time of recursive algorithms

- How do we measure the runtime of an algorithm?
  - Worst-case analysis
  - Asymptotic Analysis

Worst-case analysis

Sorting a sorted list
should be fast!!

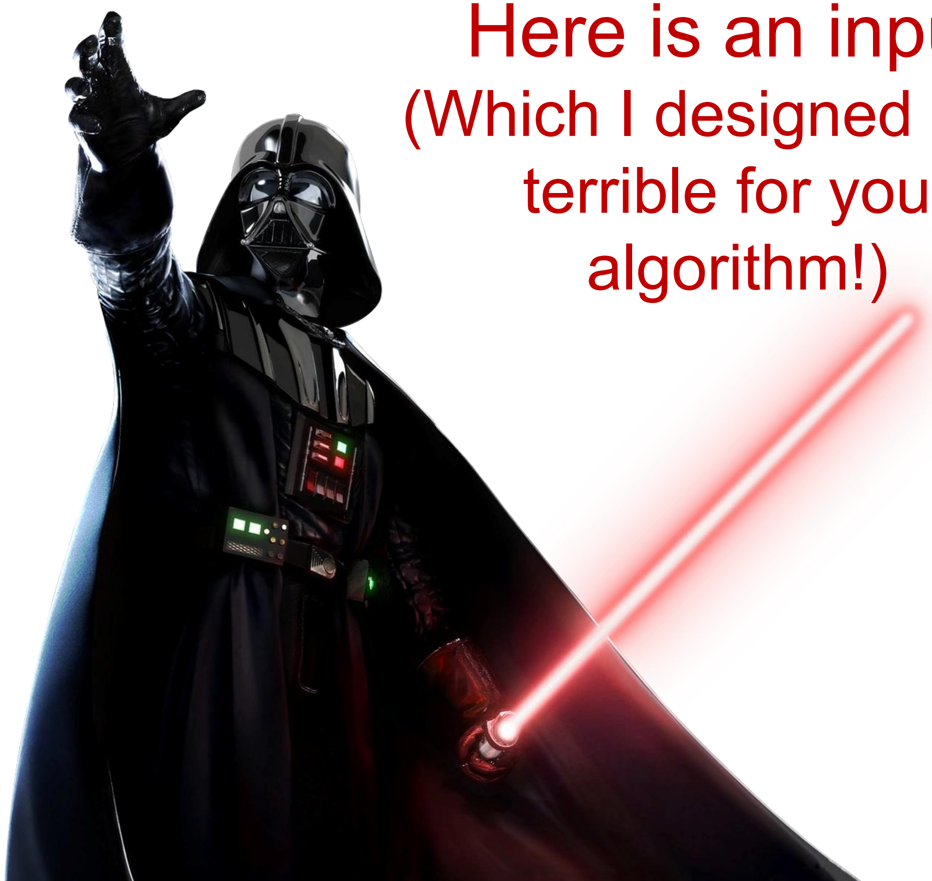
The "running time" for an algorithm is its running time on the worst possible input.



Algorithm
designer

Here is your algorithm!

Algorithm:
  Do the thing
  Do the stuff
  Return the answer



Here is an input!
(Which I designed to be terrible for your algorithm!)

Big-O notation

- What do we mean when we measure runtime?
  - We probably care about wall time: how long does it take to solve the problem, in seconds or minutes or hours?

- This is heavily dependent on the programming language, architecture, etc.
- These things are very important, but are not the point of this class.
- We want a way to talk about the running time of an algorithm, independent of these considerations.

Main idea:

Focus on how the runtime **scales** with n (the input size).

Informally….

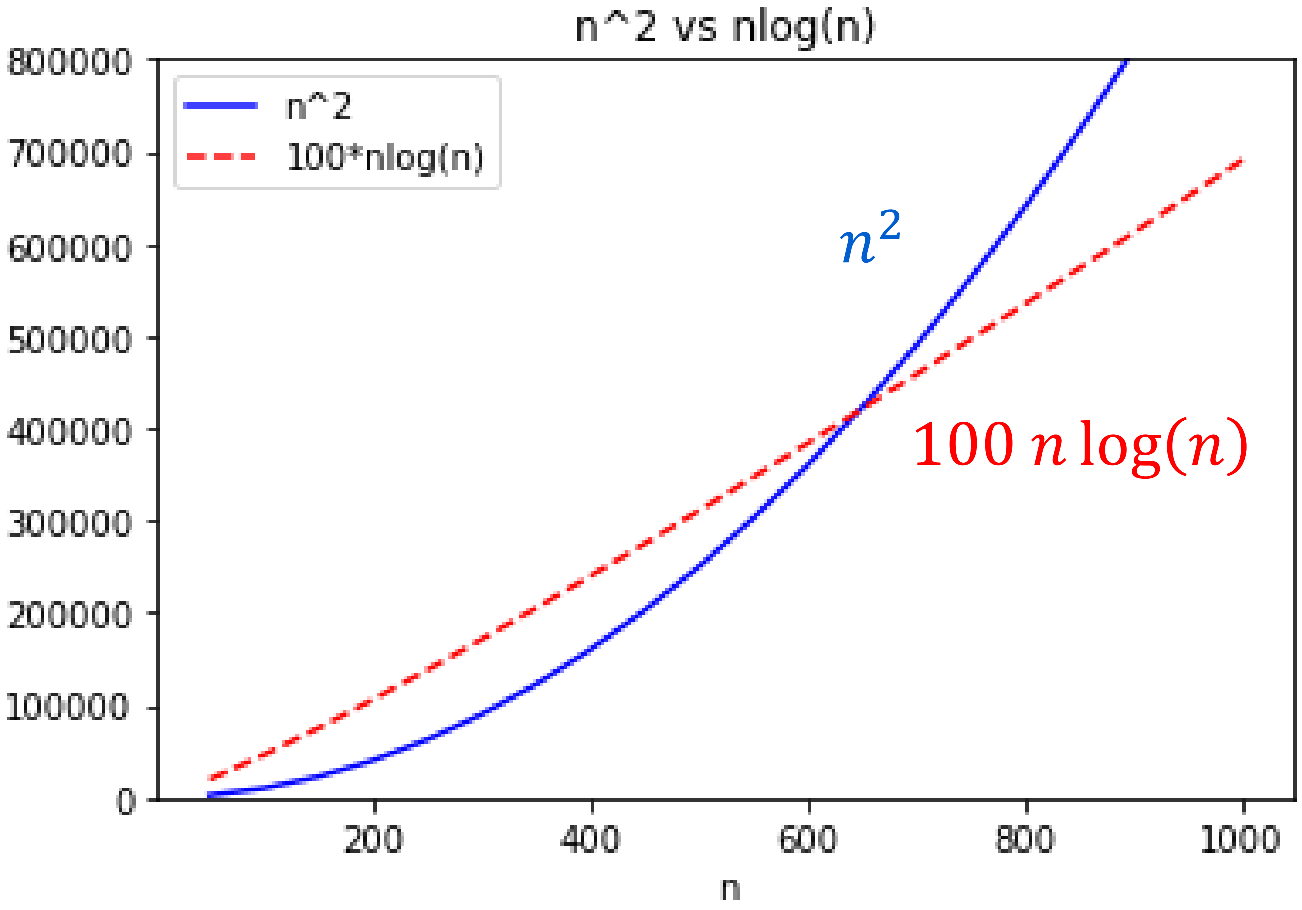
Number of operations	Asymptotic Running Time
$\frac{1}{10} n^2 + 100$	$O(n^2)$
$0.063 \cdot n^2 - .5\, n + 12.7$	$O(n^2)$
$100 \cdot n^{1.5} - 10^{10000}\sqrt{n}$	$O(n^{1.5})$
$11 \cdot n\log(n) + 1$	$O(n\log(n))$

(Only pay attention to the largest function of n that appears.)

We say this algorithm is “asymptotically faster” than the others.

So 100 n log(n) operations is "better" than n^2 operations?

But when n=200, that's not true at all!



Yeah, but it's true once n is at least 700 or so.

Asymptotic Analysis

One algorithm is "faster" than another if its runtime scales better with the size of the input.

Pros:

- Abstracts away from hardware- and language-specific issues.
- Makes algorithm analysis much more tractable.

Cons:

- Only makes sense if n is large (compared to the constant factors).

1000000000 n
is "better" than n^2 ?!?!

O(…) means an upper bound

pronounced "big-oh of …" or sometimes "oh of …"

- Let $T(n)$, $g(n)$ be functions of positive integers.
  - Think of $T(n)$ as a runtime: positive and increasing in n.
- We say "$T(n)$ is $O\left(g(n)\right)$" if $T(n)$ grows no faster than $g(n)$ as n gets large.
- Formally,

$$T(n) = O\left(g(n)\right)$$

$$\Longleftrightarrow$$

$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$

$$0 \leq T(n) \leq c \cdot g(n)$$

Example

$$2n^2 + 10 = O(n^2)$$

$$T(n) = O\big(g(n)\big)$$
$$\Longleftrightarrow$$
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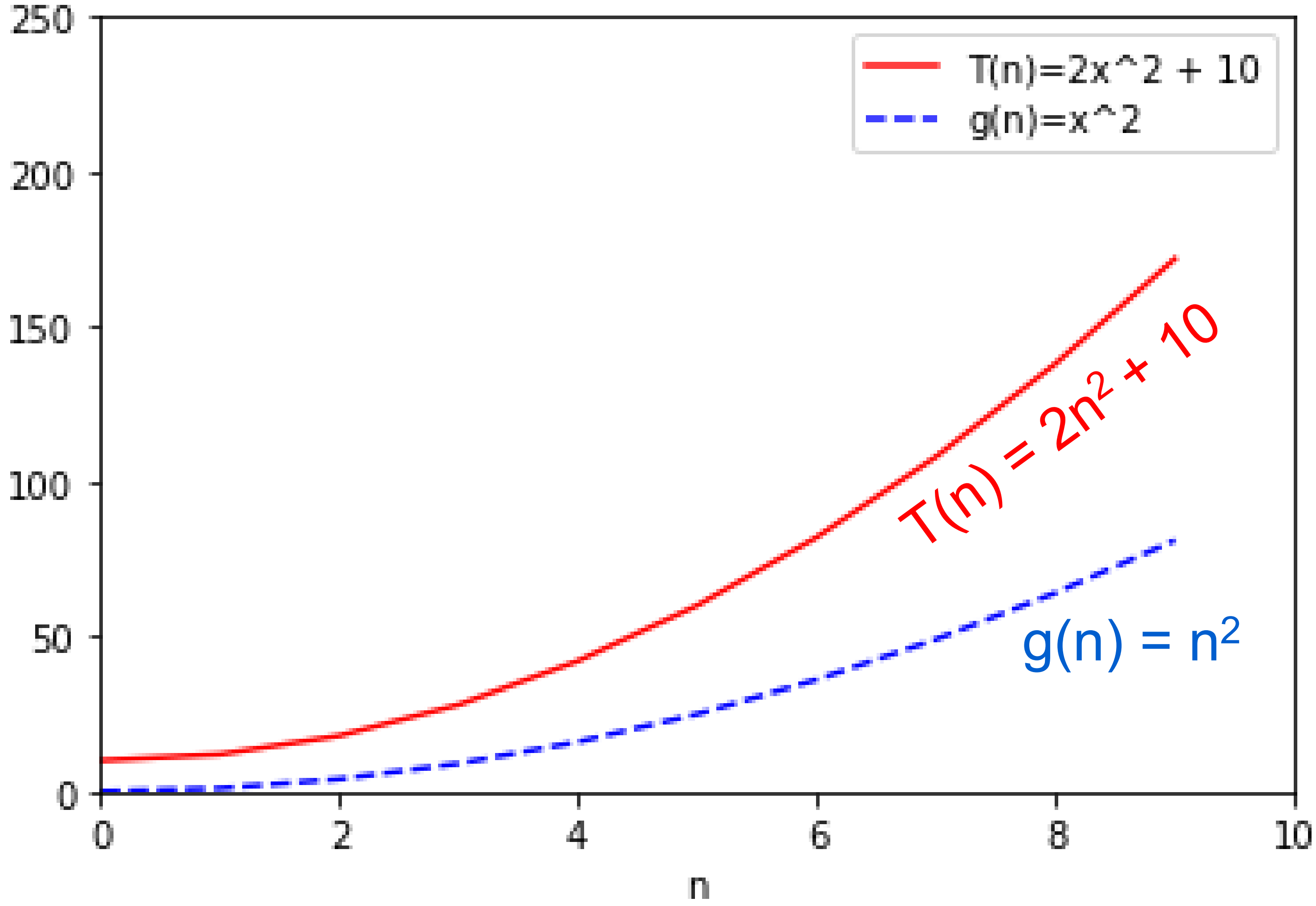

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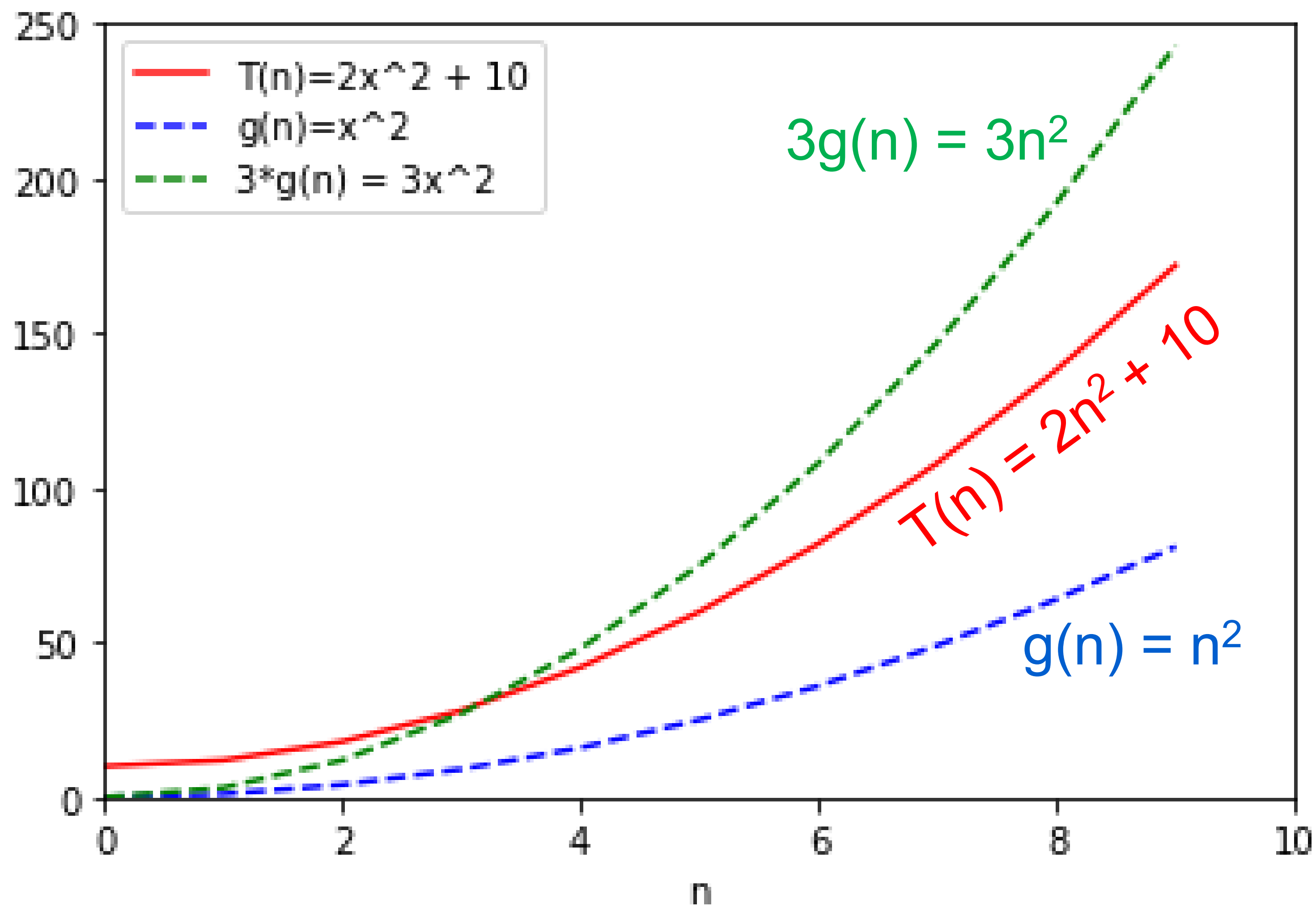

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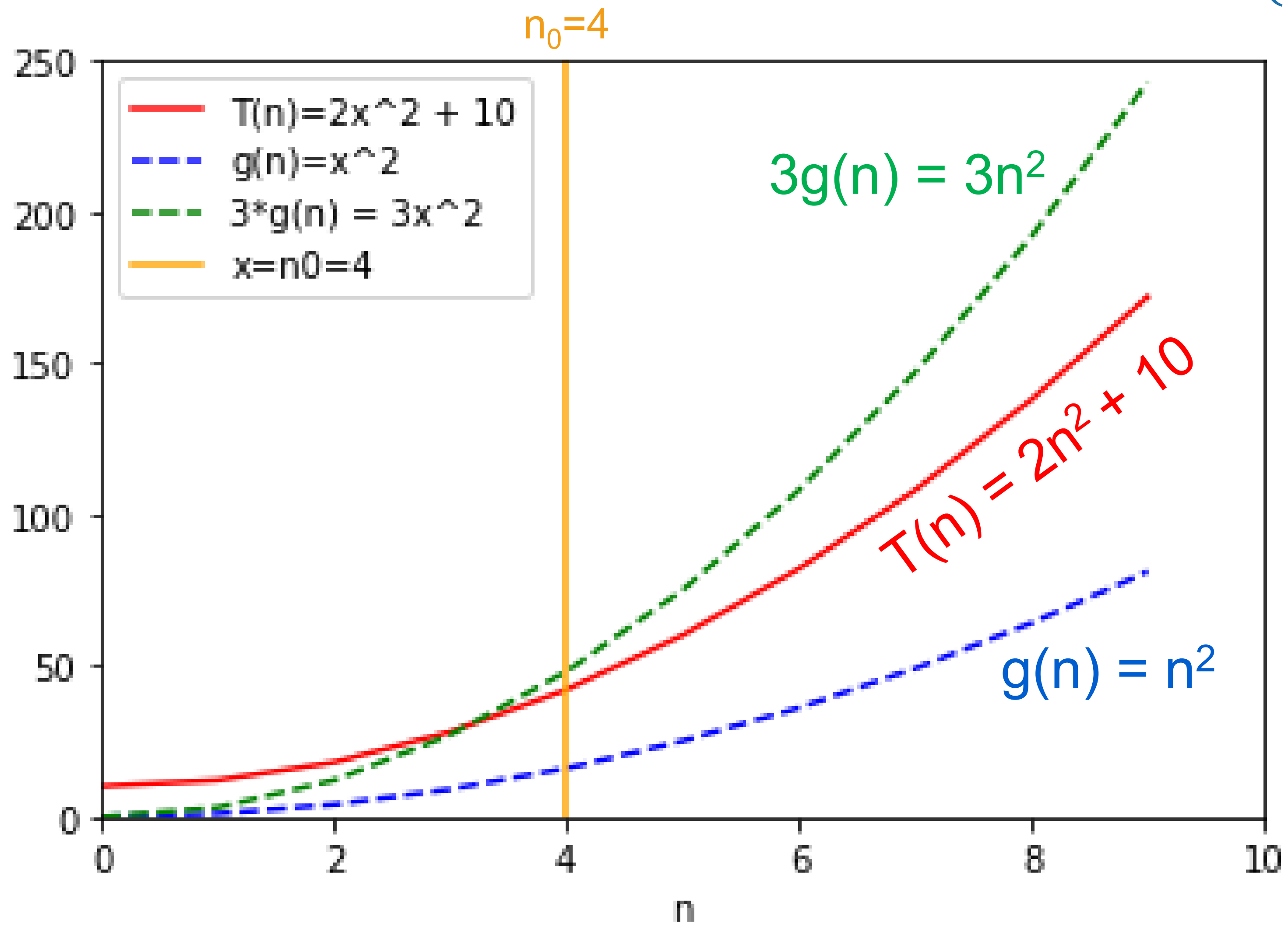

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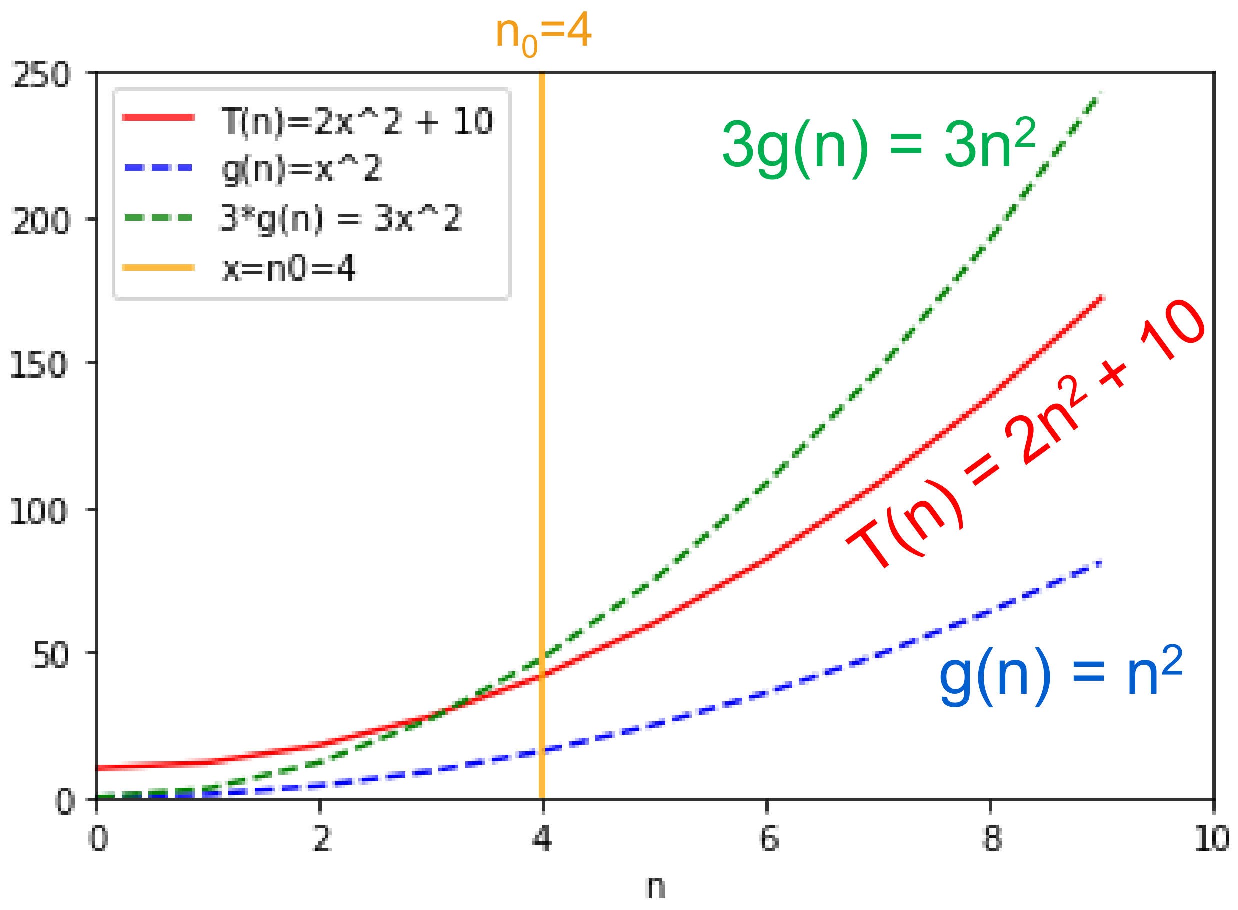


Formally:

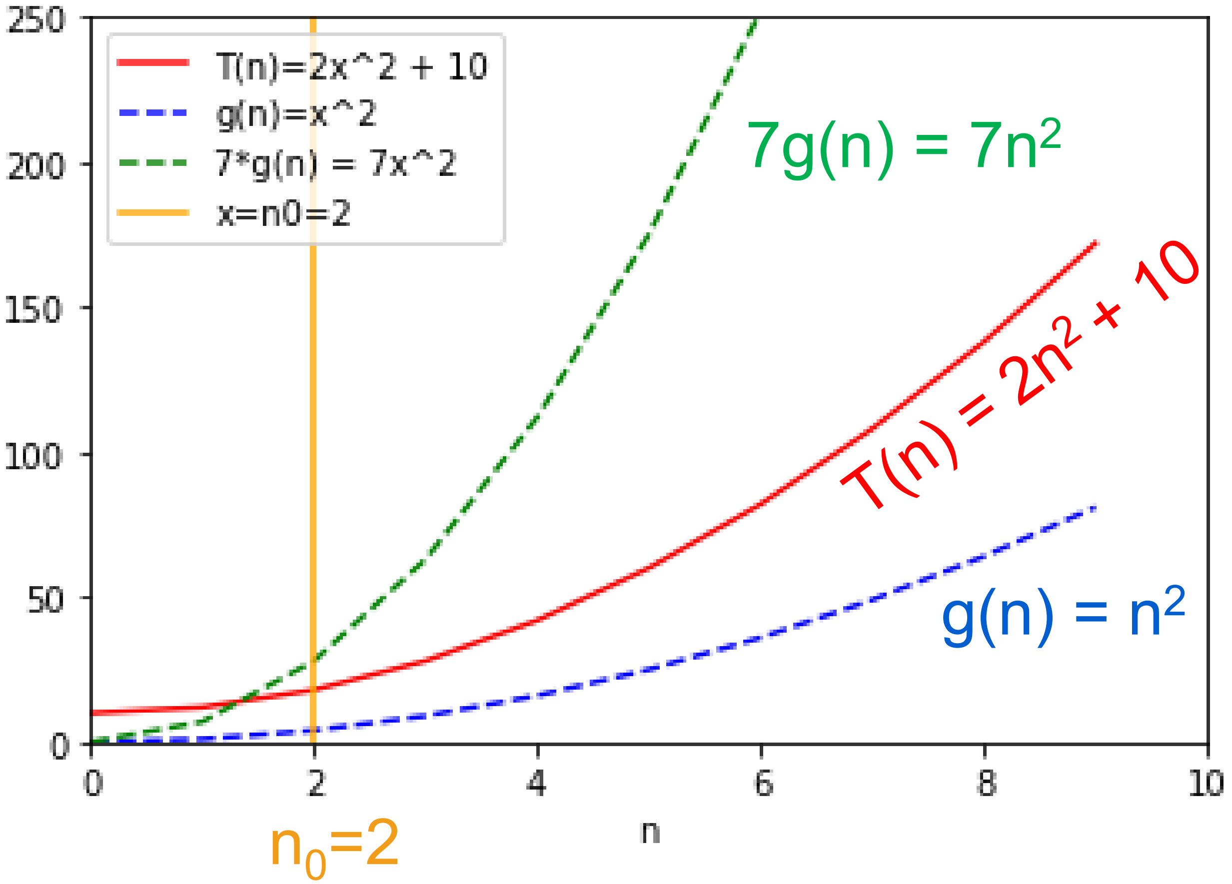
- Choose c = 3
- Choose $n_0 = 4$
- Then:

$$\forall n \geq 4,$$
$$0 \leq 2n^2 + 10 \leq 3 \cdot n^2$$

Example

$$2n^2 + 10 = O(n^2)$$

$$T(n) = O(g(n))$$
$$\Longleftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot g(n)$$



Formally:

- Choose c = 7
- Choose n_0 = 2
- Then:

$$\forall n \geq 2,$$

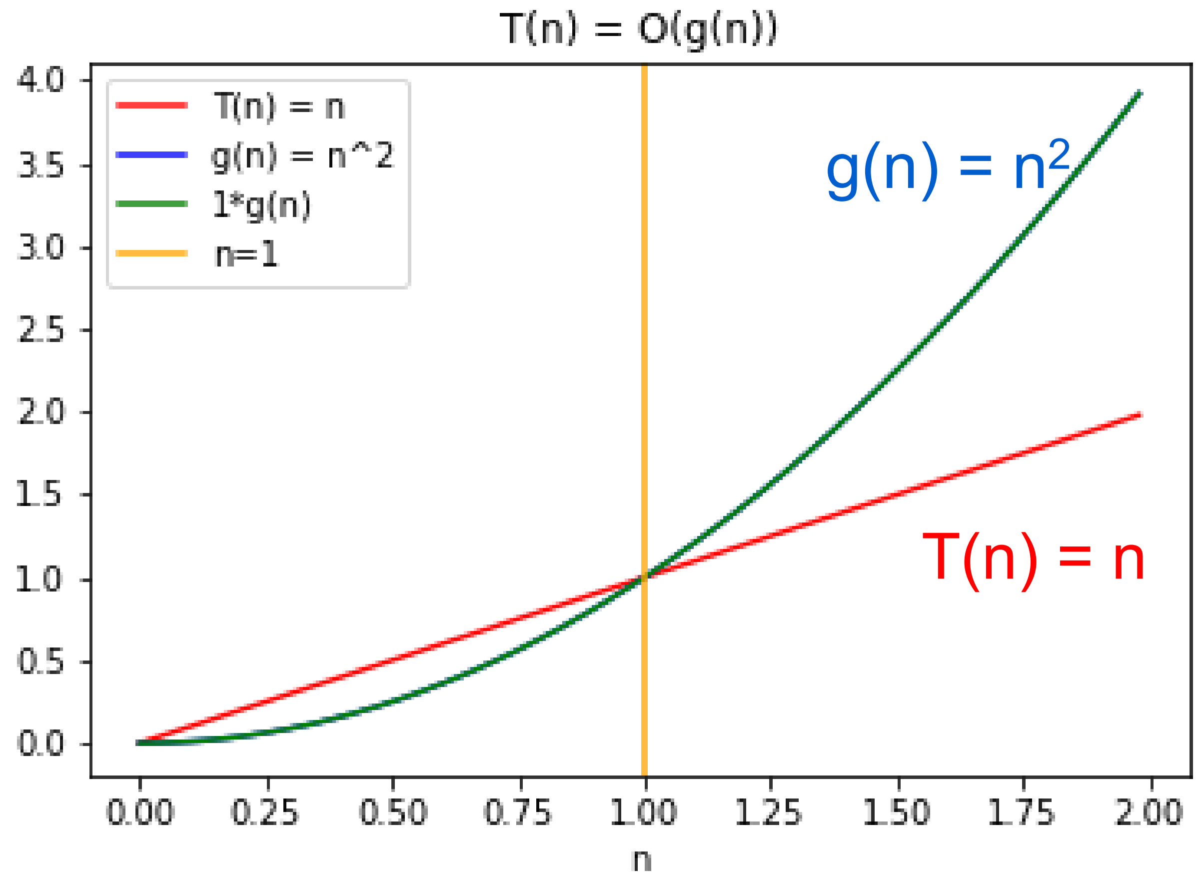
$$0 \leq 2n^2 + 10 \leq 7 \cdot n^2$$

There is not a "unique" choice of c and n_0

Another example:

$$n = O(n^2)$$

$$T(n) = O(g(n))$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \ \ s.t. \ \ \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot g(n)$$



- Choose c = 1
- Choose n_0 = 1
- Then

$$\forall n \geq 1,$$
$$0 \leq n \leq n^2$$

This is not tight bound as $n = O(n)$

Ω(…) means a lower bound

- We say "$T(n)$ is $\Omega(g(n))$" if $T(n)$ grows at least as fast as $g(n)$ as n gets large.
- Formally,

$$T(n) = \Omega(g(n))$$

$$\Leftrightarrow$$

$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$

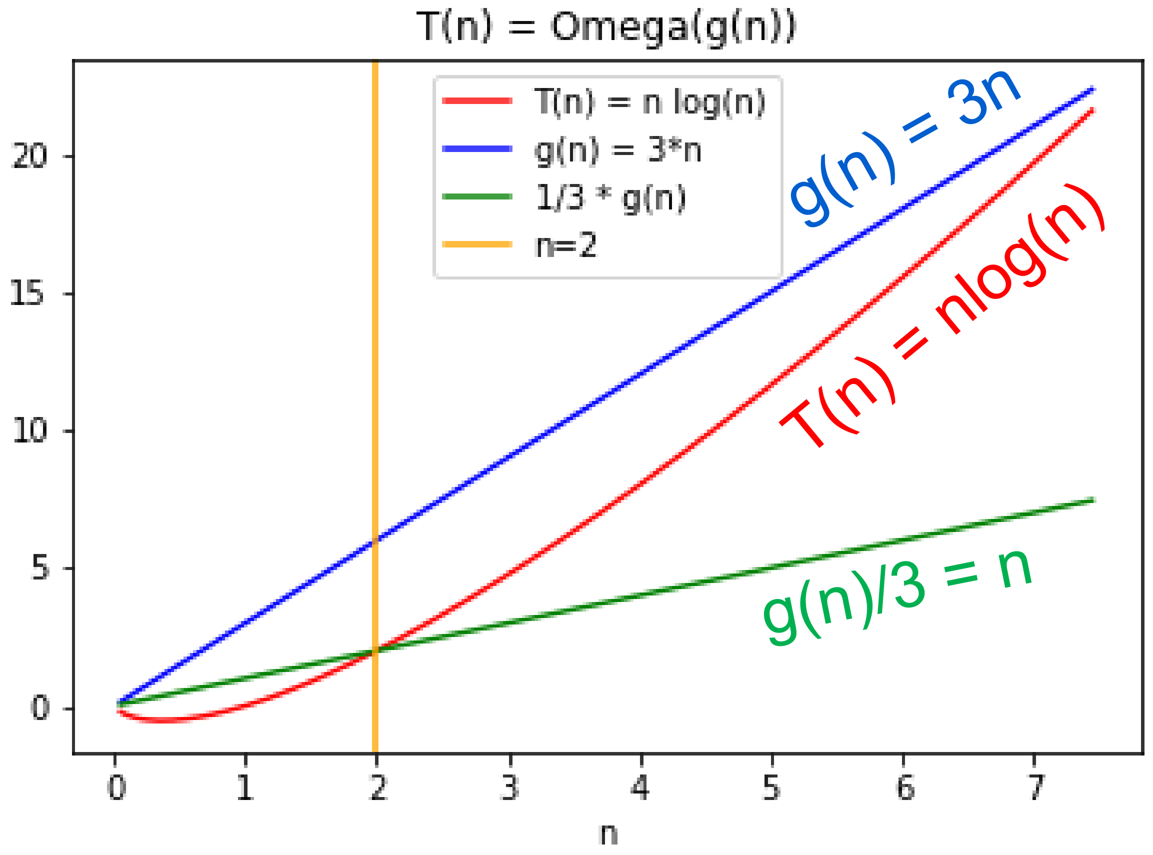
$$0 \leq c \cdot g(n) \leq T(n)$$

Switched these!!

Example

$$T(n) = \Omega(g(n))$$
$$\Longleftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq c \cdot g(n) \leq T(n)$$

$$n \log_2(n) = \Omega(3n)$$



- Choose c = 1/3
- Choose $n_0 = 2$
- Then

$$\forall n \geq 2,$$
$$0 \leq \frac{3n}{3} \leq n \log_2(n)$$

Θ(…) means both!

- We say "$T(n)$ is $\Theta(g(n))$" iff both:

$$T(n) = O\big(g(n)\big)$$

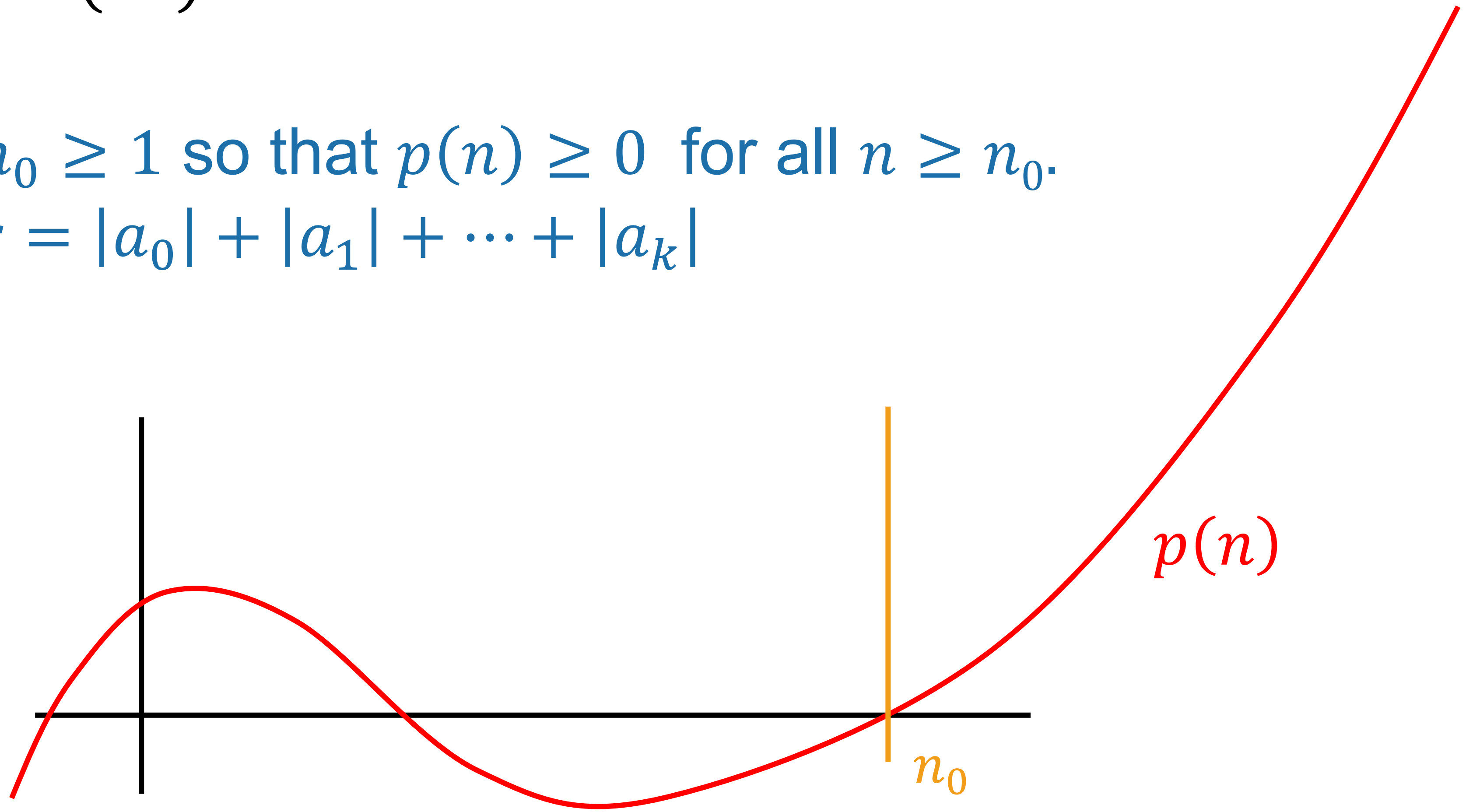
and

$$T(n) = \Omega\big(g(n)\big)$$

Example: polynomials

- Suppose the p(n) is a polynomial of degree k:

  $p(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k$ where $a_k > 0$.
- Then $p(n) = O(n^k)$
- Proof:
  - Choose $n_0 \geq 1$ so that $p(n) \geq 0$ for all $n \geq n_0$.
  - Choose $c = |a_0| + |a_1| + \cdots + |a_k|$



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- Then $p(n) = O\left(n^k\right)$
- Proof:
  - Choose $n_0 \geq 1$ so that $p(n) \geq 0$ for all $n \geq n_0$.
  - Choose $c = |a_0| + |a_1| + \cdots + |a_k|$
  - Then for all $n \geq n_0$:
  - $0 \leq p(n) = |p(n)| \leq |a_0| + |a_1|n + \cdots + |a_k|n^k$
  - $\leq |a_0|n^k + |a_1|n^k + \cdots + |a_k|n^k$
  - $= c \cdot n^k$

Definition of c

Because $n \leq n^k$ for $n \geq n_0 \geq 1$.

Example: more polynomials

- For any $k \geq 1, n^k$ is NOT $O(n^{k-1})$.
- Proof:
  - Suppose that it were.
    - Then there is some c, n_0 so that $n^k \leq c \cdot n^{k-1}$ for all $n \geq n_0$
  - Aka, $n \leq c$ for all $n \geq n_0$
  - But that's not true!
  - We have a contradiction!
    - It *can't* be that $n^k = O(n^{k-1})$.

Take-away from examples

- To prove $T(n) = O(g(n))$, you have to come up with c and n_0 so that the definition is satisfied.

- To prove T(n) is NOT O(g(n)), one way is **proof by contradiction**:
  - Suppose (to get a contradiction) that someone gives you a c and an n_0 so that the definition ***is*** satisfied.
  - Show that this someone must by lying to you by deriving a contradiction.

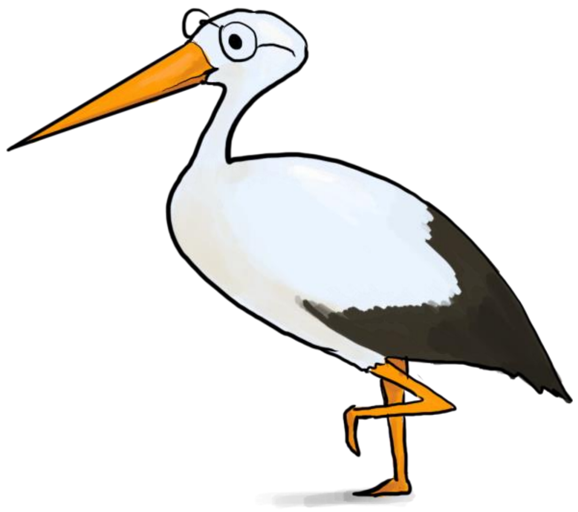
Yet more examples

- $n^3 + 3n = O(n^3 - n^2)$
- $n^3 + 3n = \Omega(n^3 - n^2)$
- $n^3 + 3n = \Theta(n^3 - n^2)$

- 3^n is **NOT** $O(2^n)$
- $\log(n) = \Omega(\ln(n))$
- $\log(n) = \Theta(2^{\log\log(n)})$

remember that $\log = \log_2$ in this class.

Work through these on your own!



Some brainteasers

- Are there functions f, g so that NEITHER $f = O(g)$ nor $f = \Omega(g)$?
- Are there non-decreasing functions f, g so that the above is true?
- Define the n'th fibonacci number by $F(0) = 1$, $F(1) = 1$, $F(n) = F(n-1) + F(n-2)$ for $n > 1$.
  - 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

True or false:

- $F(n) = O(2^n)$
- $F(n) = \Omega(2^n)$

Recurrence Relations!

Recurrence Relations!

- How do we calculate the runtime of a recursive algorithm?

Running time of MergeSort

- Let's call this running time T(n), when the input has length n.

- We know that T(n) = O(nlog(n)).

- We also know that T(n) satisfies:

$$T(n) \leq 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

Last time we showed that the time to run MERGE on a problem of size n is at most c*n operations.

```
MERGESORT(A):
    n = length(A)
    if n ≤ 1:
        return A
    L = MERGESORT(A[1:n/2-1])
    R = MERGESORT(A[n/2:n])
    return MERGE(L,R)
```

Recurrence Relations

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$ is a **recurrence relation.**
- It gives us a formula for T(n) in terms of T(less than n)

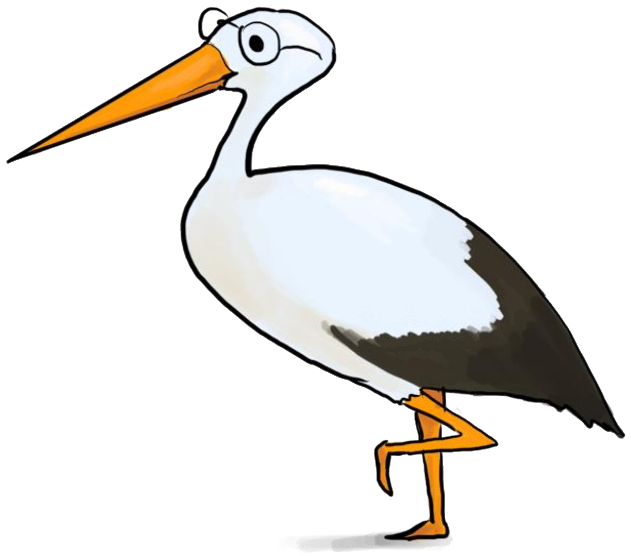
- The challenge:
  Given a recurrence relation for T(n), find a closed form expression for T(n).

- For example, T(n) = O(nlog(n)) in this case

Technicalities I: Base Case

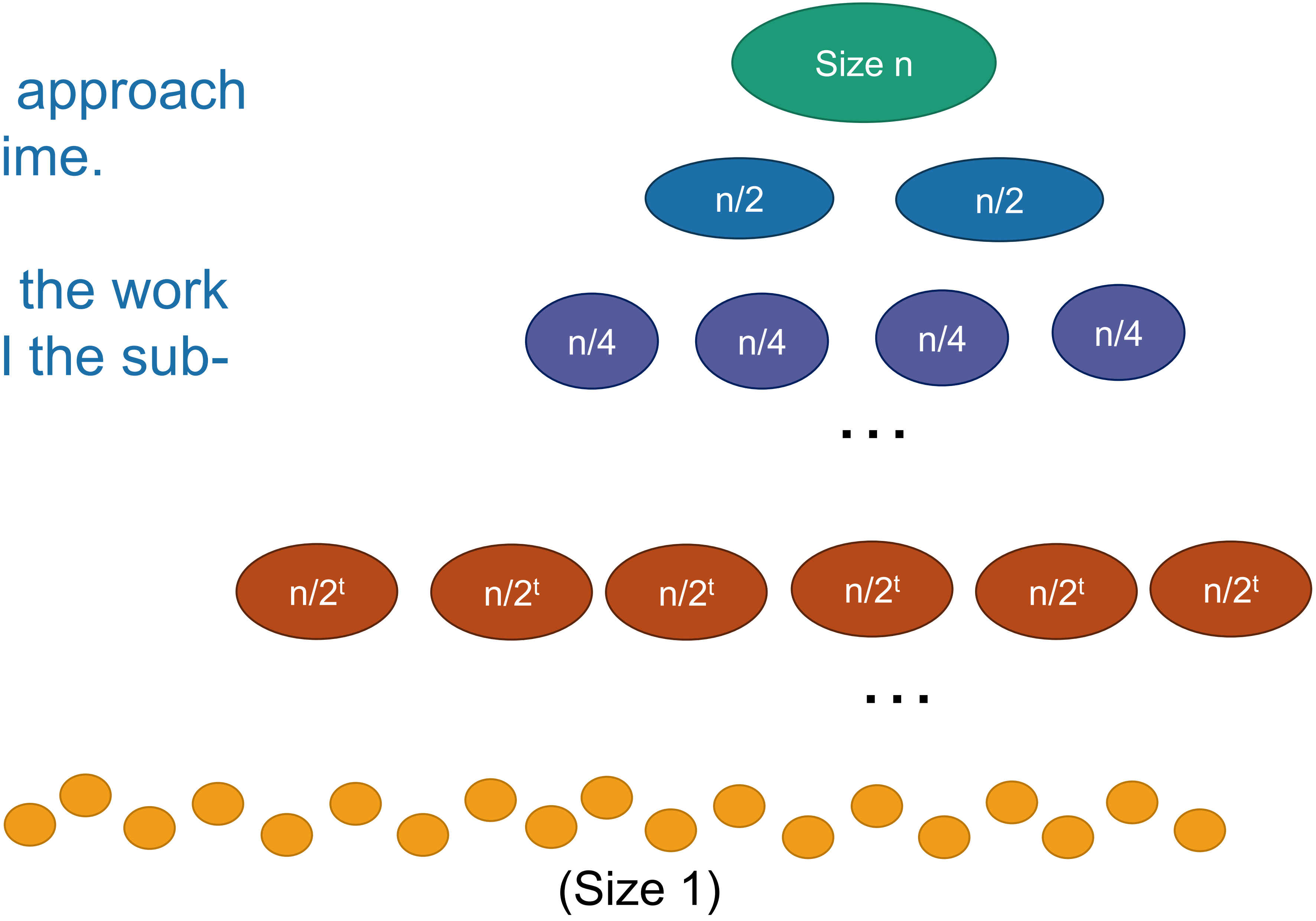
- Formally, we should always have base cases with recurrence relations.
- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$ with $T(1) = O(1)$

Why does T(1) = O(1)?



One approach

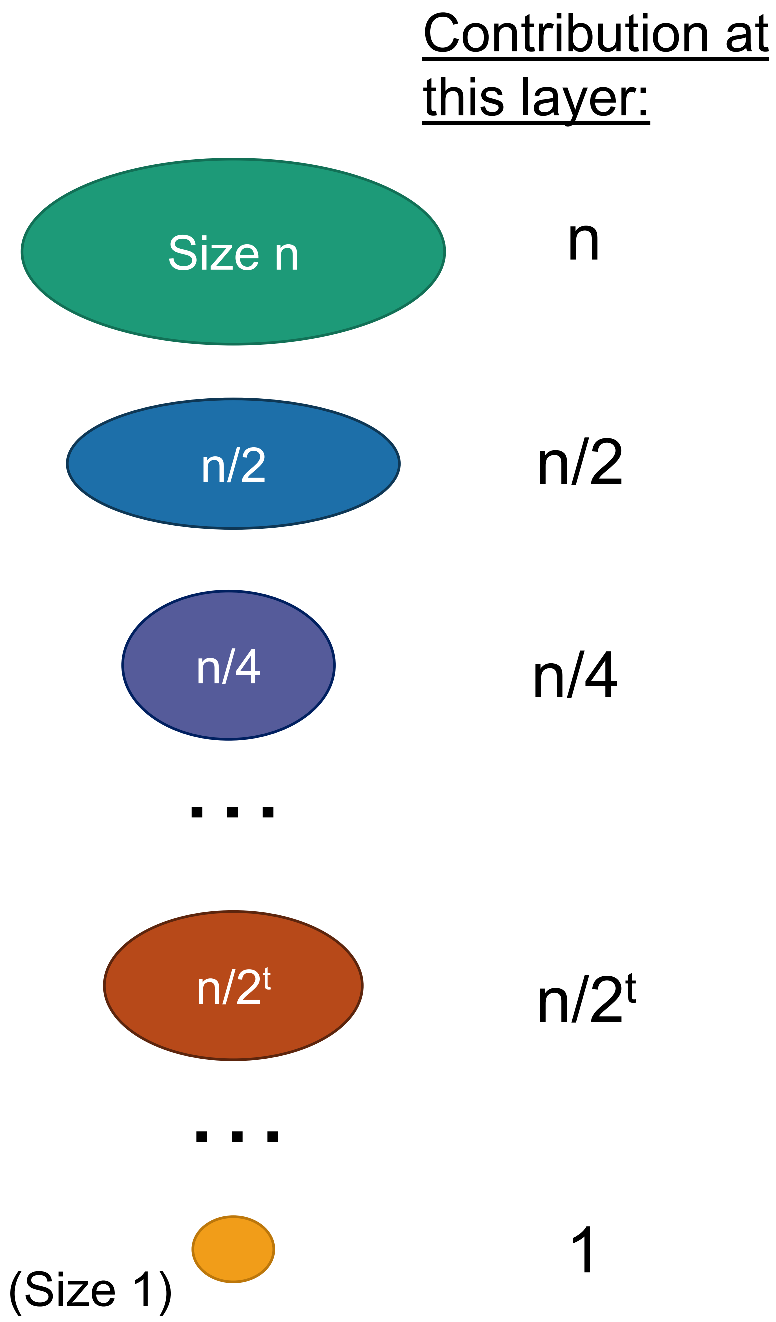
- The “tree” approach from last time.
- Add up all the work done at all the sub-problems.



Another Example

- $T_1(n) = T_1\left(\frac{n}{2}\right) + n, \quad T_1(1) = 1.$
- Adding up over all layers:

$$\sum_{i=0}^{\log(n)} \frac{n}{2^i} = 2n - 1$$



Aside

Finite Geometric Series

To find the sum of a finite geometric series, use the formula,

$$S_n = \frac{a_1(1-r^n)}{1-r}, r \neq 1,$$

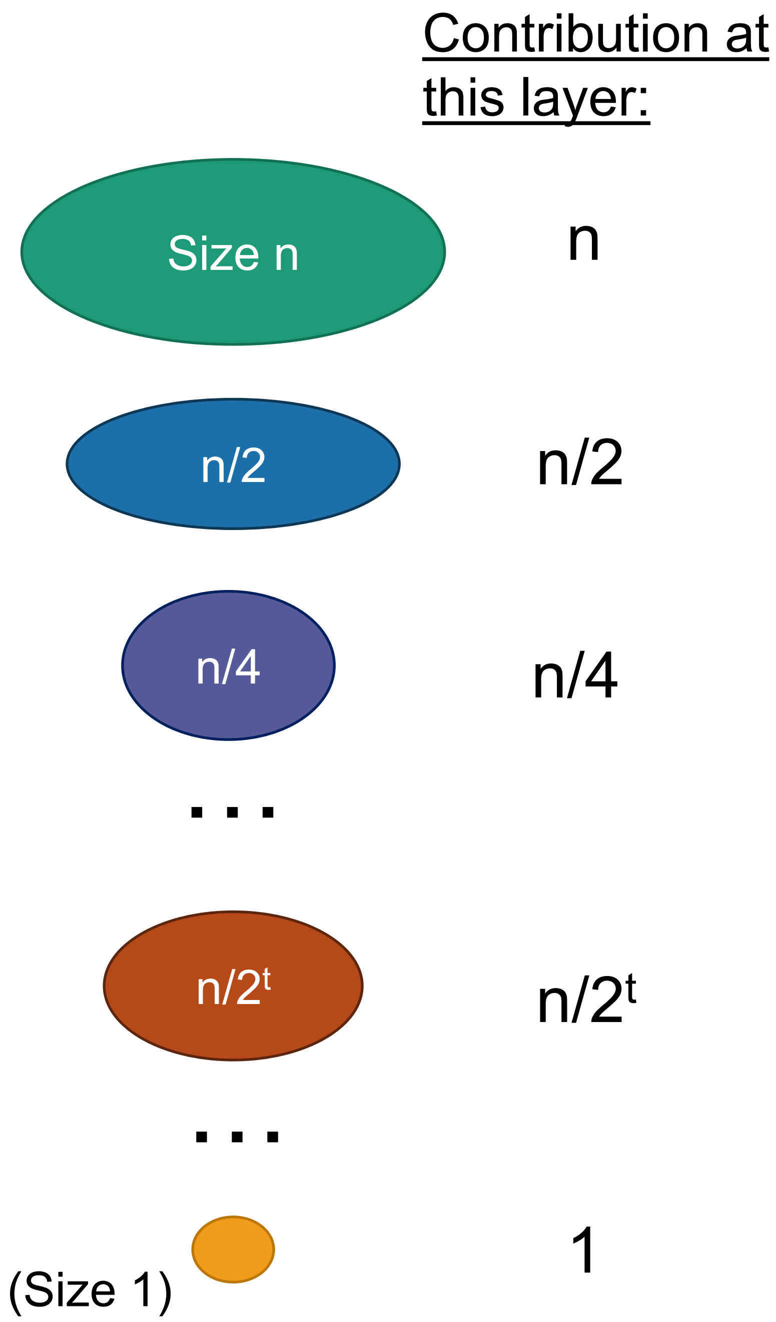
where n is the number of terms, a_1 is the first term and r is the common ratio .

Another Example

- $T_1(n) = T_1\left(\frac{n}{2}\right) + n, \quad T_1(1) = 1.$
- Adding up over all layers:

$$\sum_{i=0}^{\log(n)} \frac{n}{2^i} = 2n - 1$$

So $T_1(n) = O(n)$.

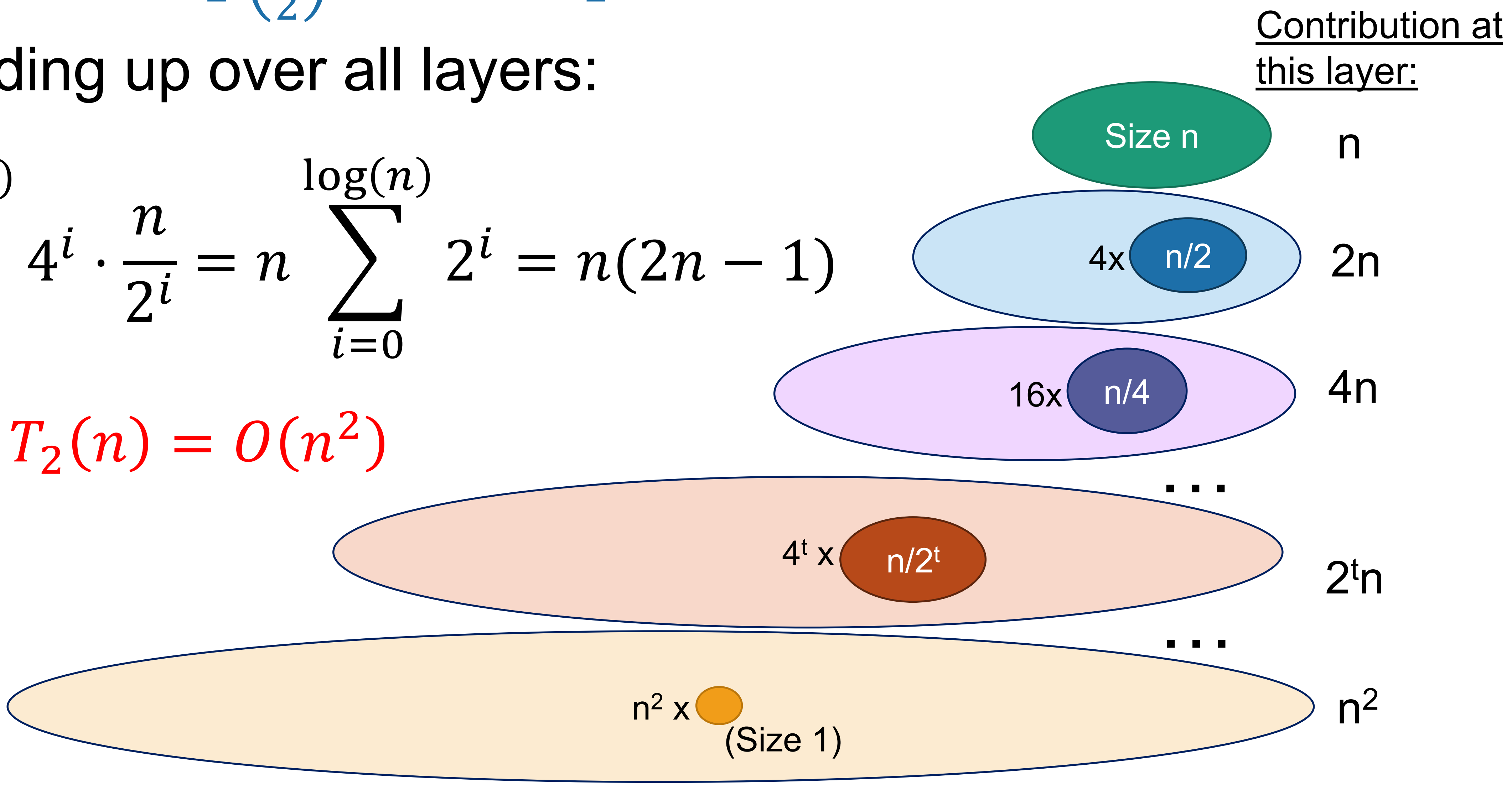


Another Example

- $T_2(n) = 4T_2\left(\frac{n}{2}\right) + n, \qquad T_2(1) = 1.$
- Adding up over all layers:

$$\sum_{i=0}^{\log(n)} 4^i \cdot \frac{n}{2^i} = n \sum_{i=0}^{\log(n)} 2^i = n(2n-1)$$

- So $T_2(n) = O(n^2)$



More examples

T(n) = time to solve a problem of size n.

Recursion 1

- T(n) = 4 T(n/2) + O(n)
- T(n) = O(n^2)

Recursion 2

- T(n) = 3 T(n/2) + O(n)
- T(n) = O($n^{\log_2(3)}$ $\approx n^{1.6}$)

Recursion 3

- T(n) = 2T(n/2) + O(n)
- T(n) = O(nlog(n))

Recursion 4

- T(n) = T(n/2) + O(n)
- T(n) = O(n)

What’s the pattern?!?!?!?!

The master theorem

- A formula for many recurrence relations.

Jedi master Yoda

The master theorem (Optional)

We can also take n/b to mean either $\left\lfloor \frac{n}{b} \right\rfloor$ or $\left\lceil \frac{n}{b} \right\rceil$ and the theorem is still true.

- Suppose that $a \geq 1, b > 1,$ and d are constants (independent of n).
- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

Three parameters:

a : number of subproblems

b : factor by which input size shrinks

d : need to do n^d work to create all the subproblems and combine their solutions.

Many symbols those are….

Examples

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d).$$

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

- Recursion 1
  - $T(n) = 4\,T(n/2) + O(n)$
  - $T(n) = O(n^2)$

a = 4, b = 2, d = 1 — $a > b^d$ ✓

- Recursion 2
  - $T(n) = 3\,T(n/2) + O(n)$
  - $T(n) = O(n^{\log_2(3)} \approx n^{1.6})$

a = 3, b = 2, d = 1 — $a > b^d$ ✓

- Recursion 3
  - $T(n) = 2T(n/2) + O(n)$
  - $T(n) = O(n\log(n))$

a = 2, b = 2, d = 1 — $a = b^d$ ✓

- Recursion 4
  - $T(n) = T(n/2) + O(n)$
  - $T(n) = O(n)$

a = 1, b = 2, d = 1 — $a < b^d$ ✓

Acknowledgement

- Stanford University

Thank You