Indian Institute of Information Technology Allahabad

Data Structures and Algorithms

Single Source Shortest Paths (SSSP): Dijkstra Algo

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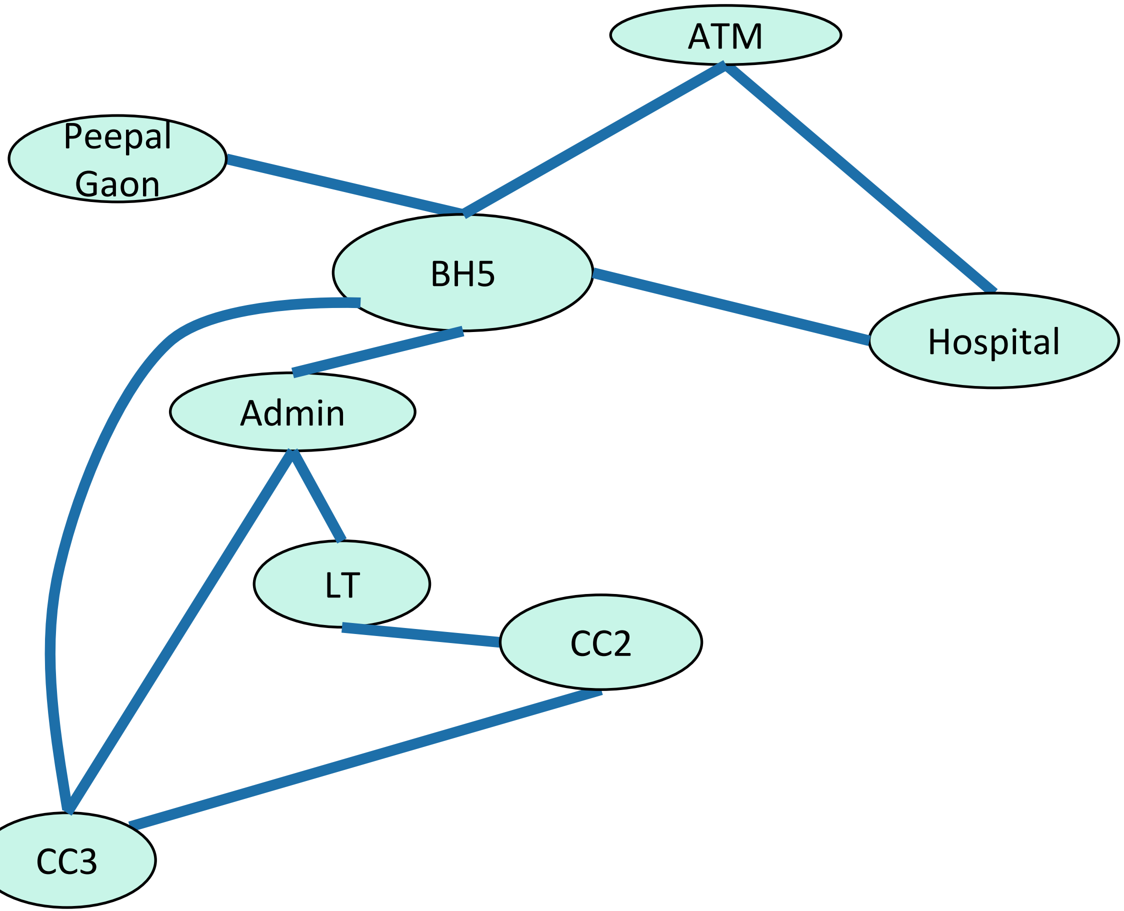
This Class

- Shortest Paths
  - BFS
  - What if the graphs are weighted?

- Single Source
  - Dijkstra!
  - Bellman-Ford!
- All Source
  - Floyd-Warshall

IIITA Graph



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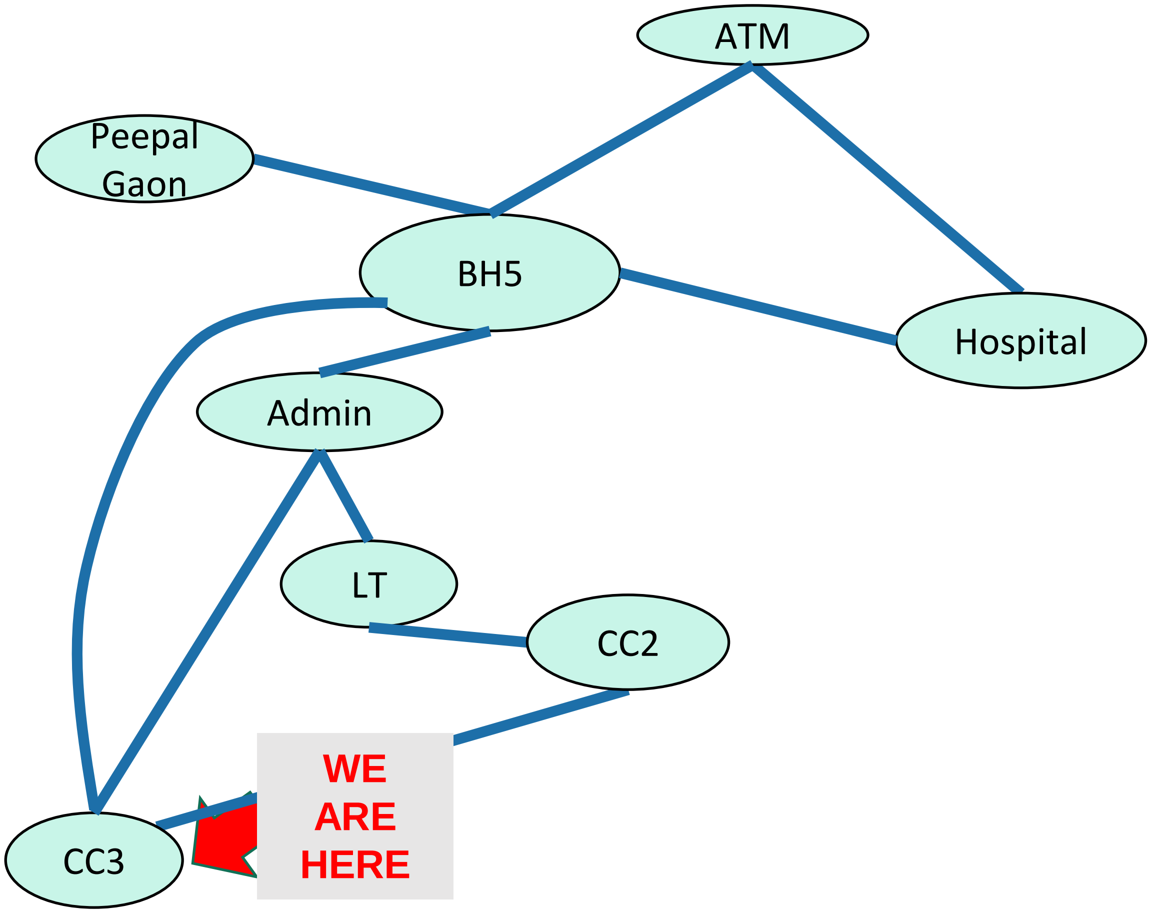
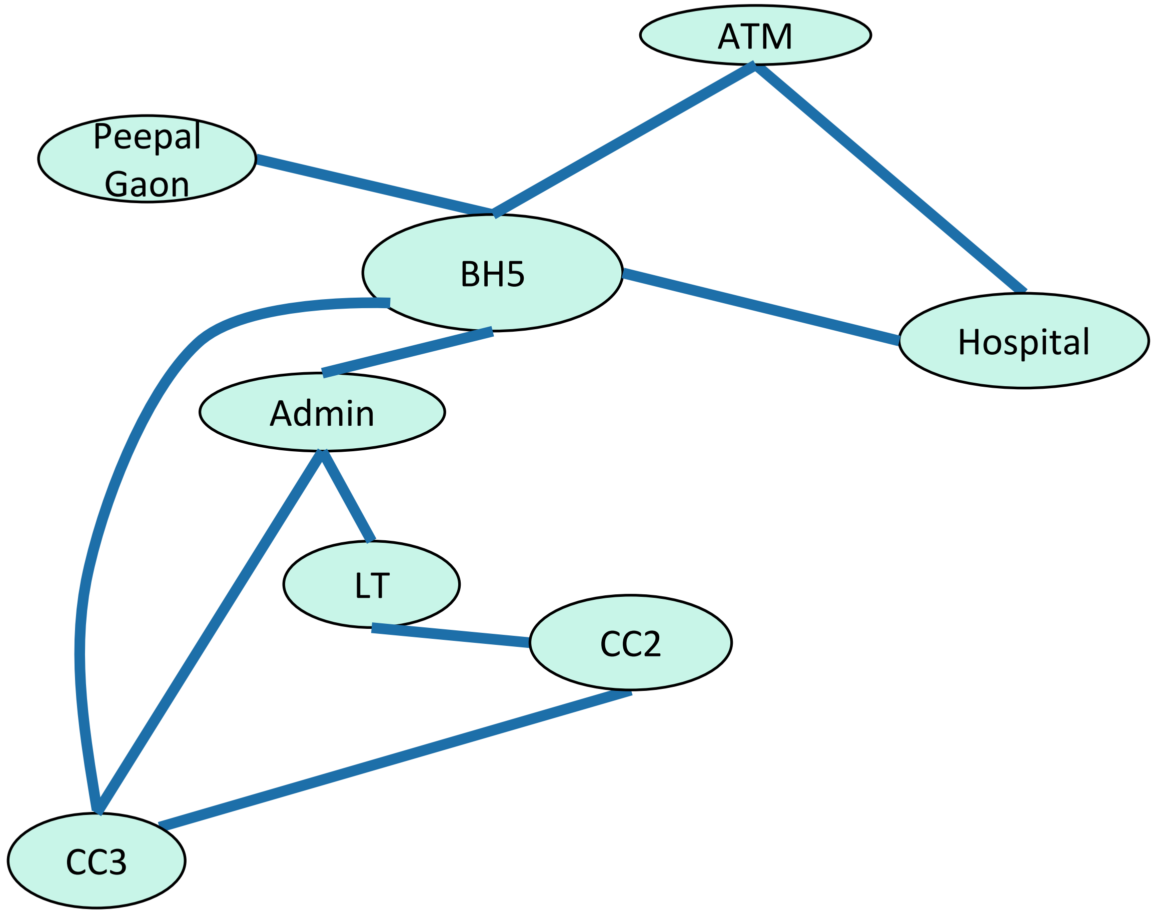


Shortest path from BH5 to CC2?



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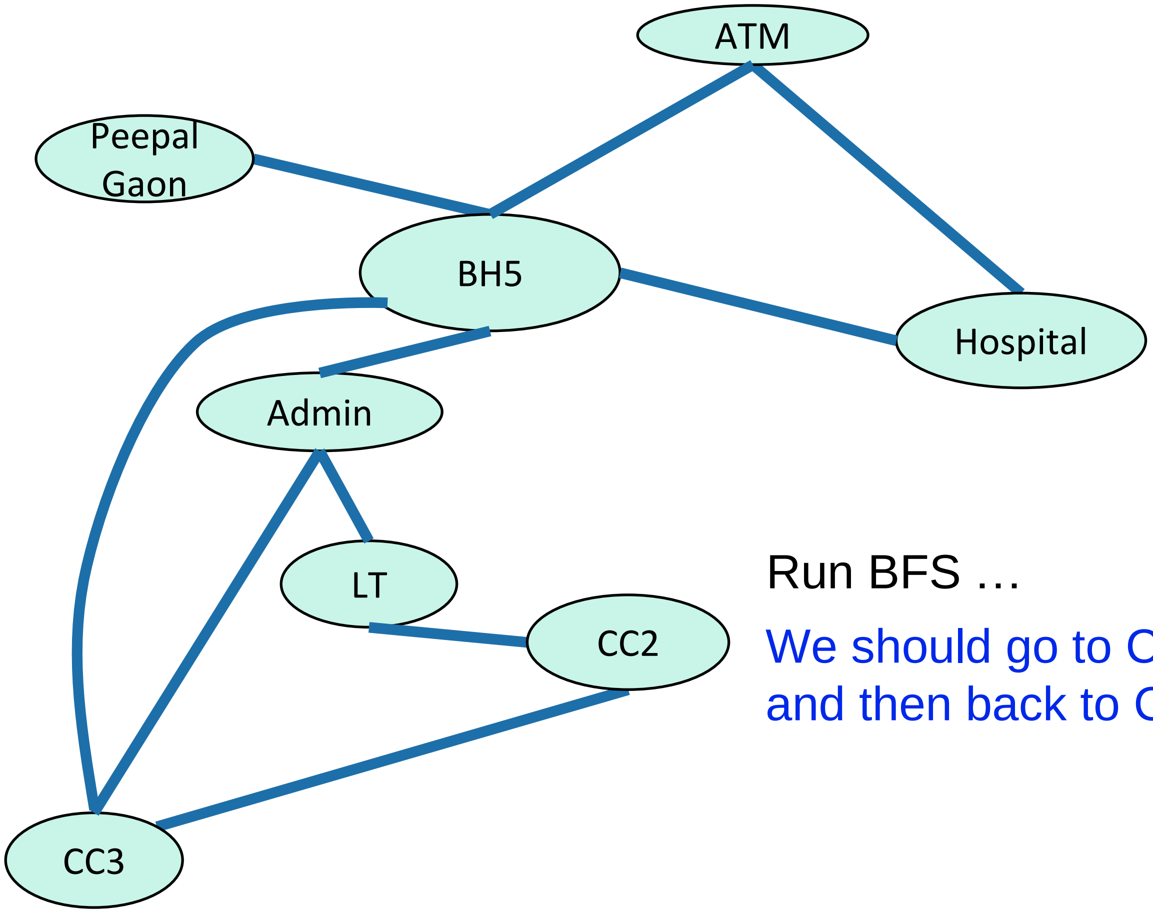

Run BFS …

We should go to CC3 and then back to CC2 !!!

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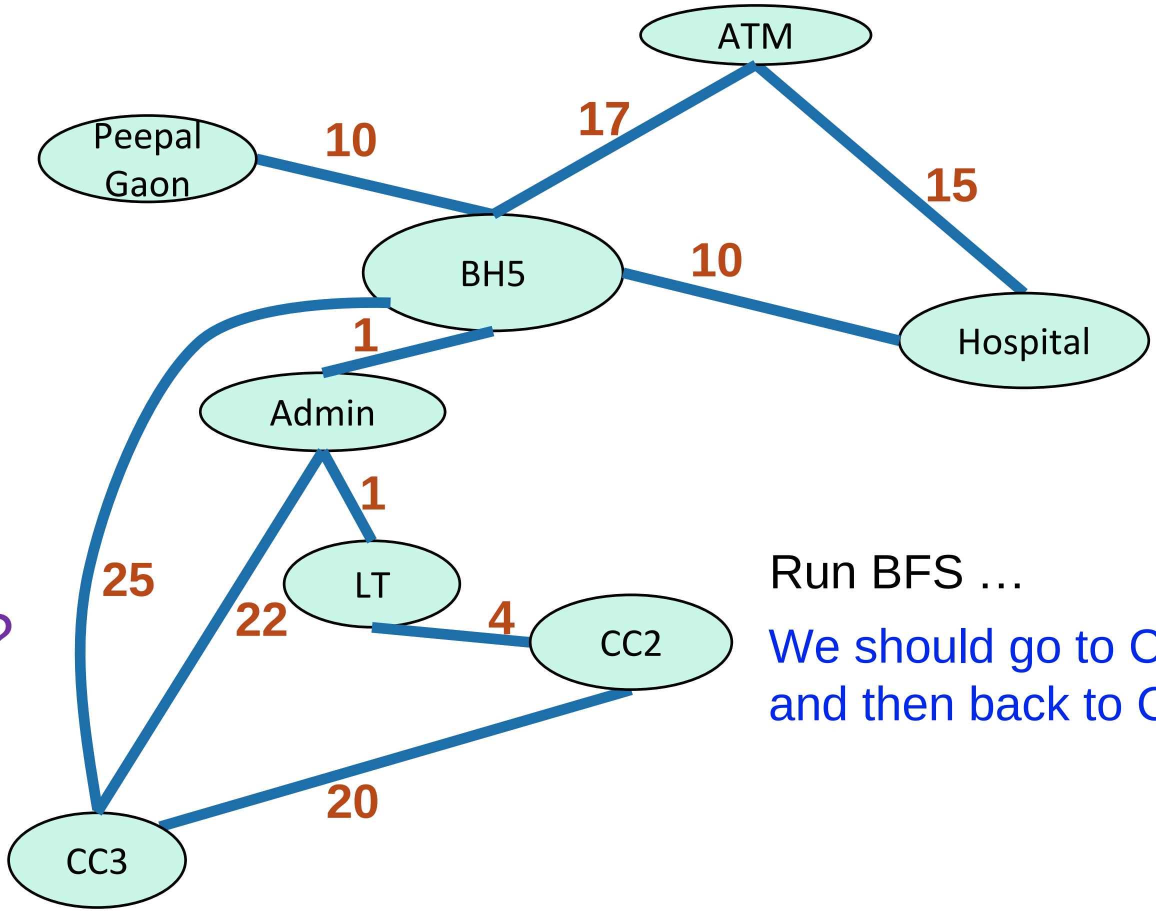

What if we label the edges by walking time ?

Run BFS …

We should go to CC3 and then back to CC2 !!!

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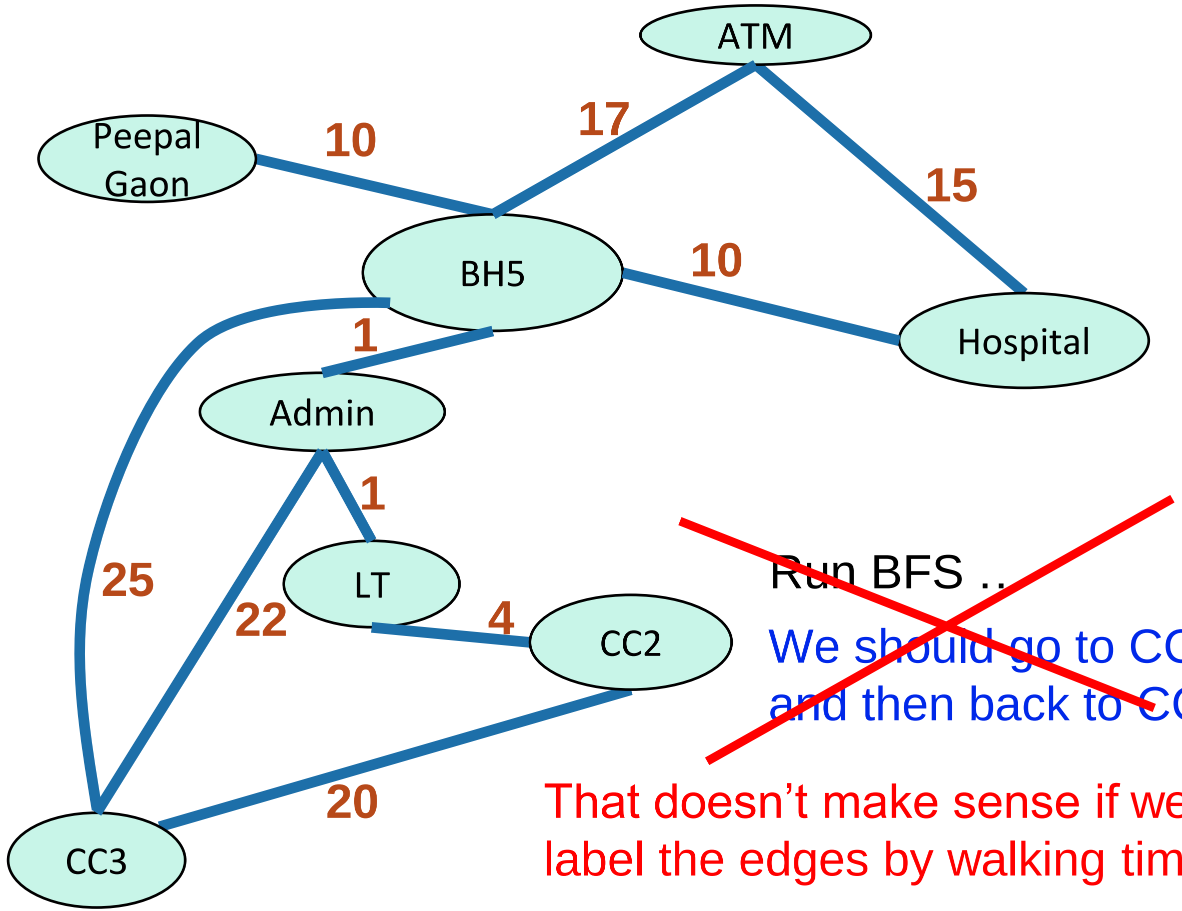


~~Run BFS ..~~

~~We should go to CC3 and then back to CC2 !!!~~

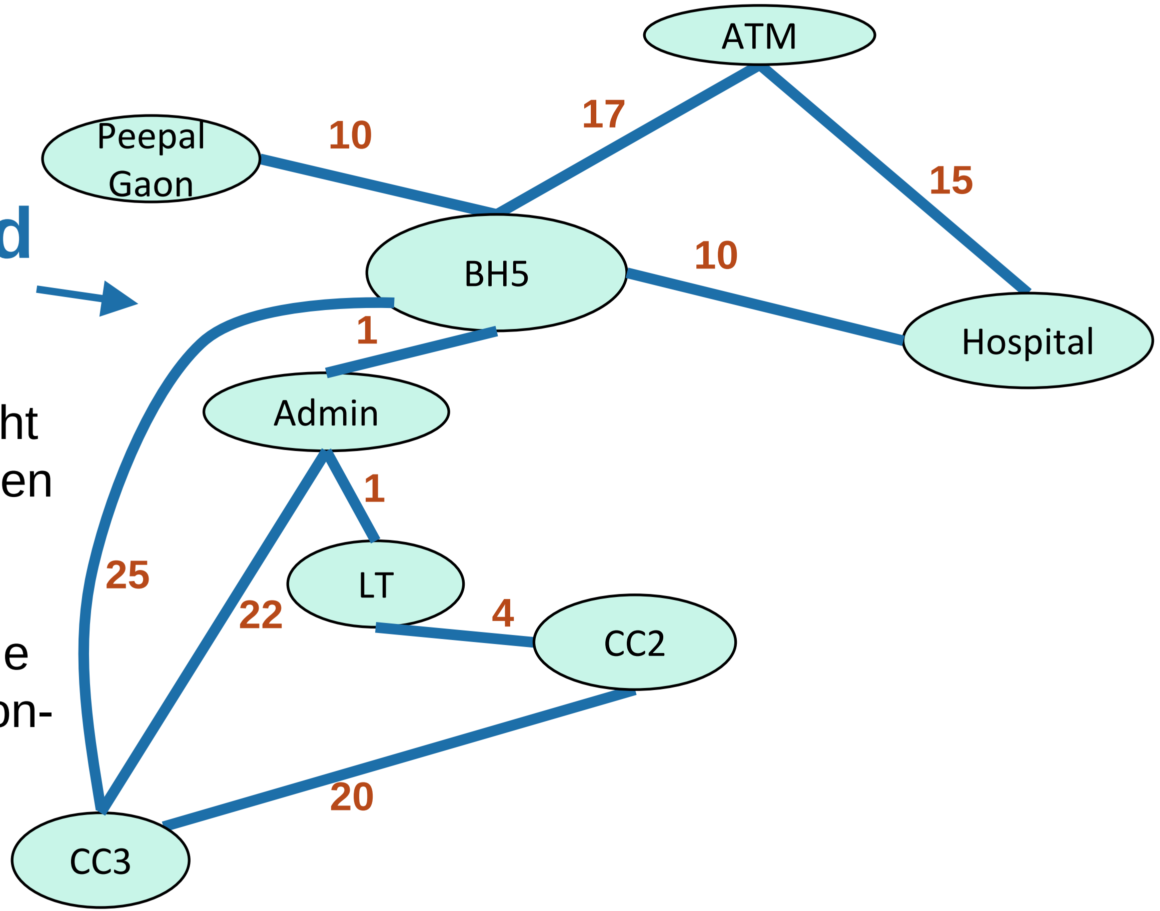
That doesn't make sense if we label the edges by walking time.

Shortest path from BH5 to CC2?

weighted graph

w(u,v) = weight of edge between u and v.

For now, edge weights are non-negative.



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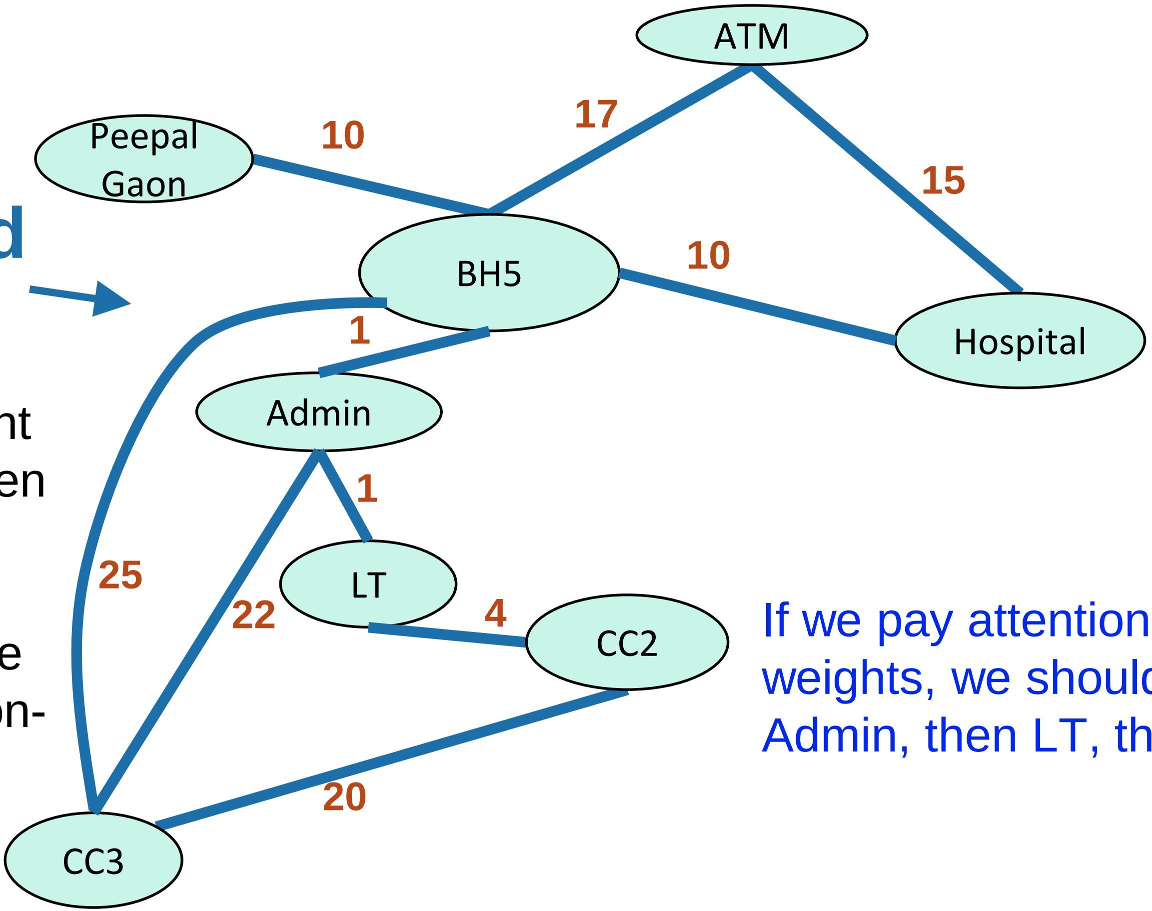


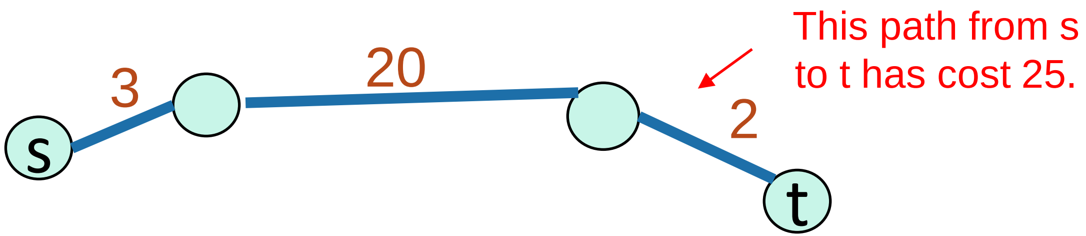
w(u,v) = weight of edge between u and v.

For now, edge weights are non-negative.

If we pay attention to the weights, we should go to the Admin, then LT, then CC2.

Shortest path problem

- What is the shortest path between u and v in a weighted graph?
  - the **cost** of a path is the sum of the weights along that path



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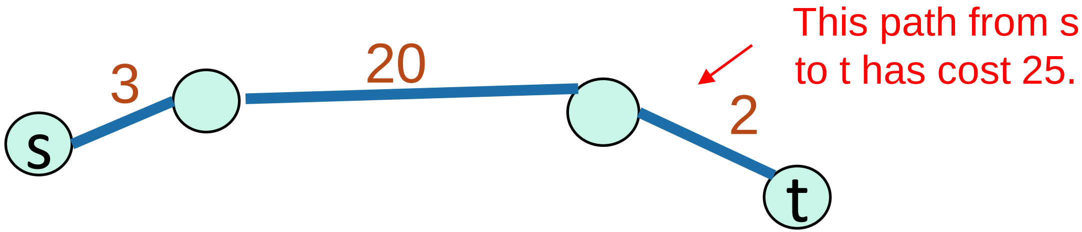

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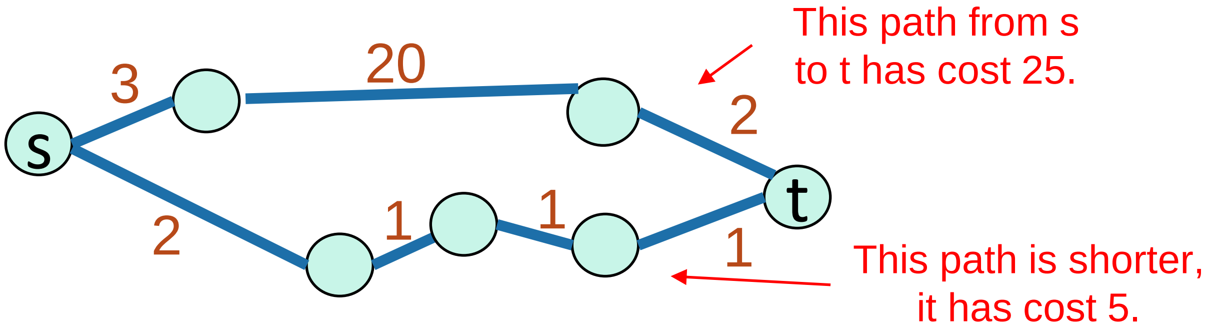

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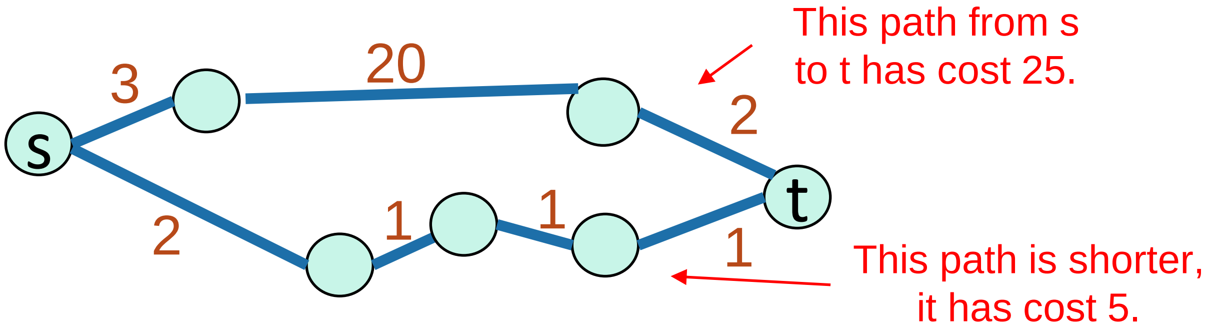
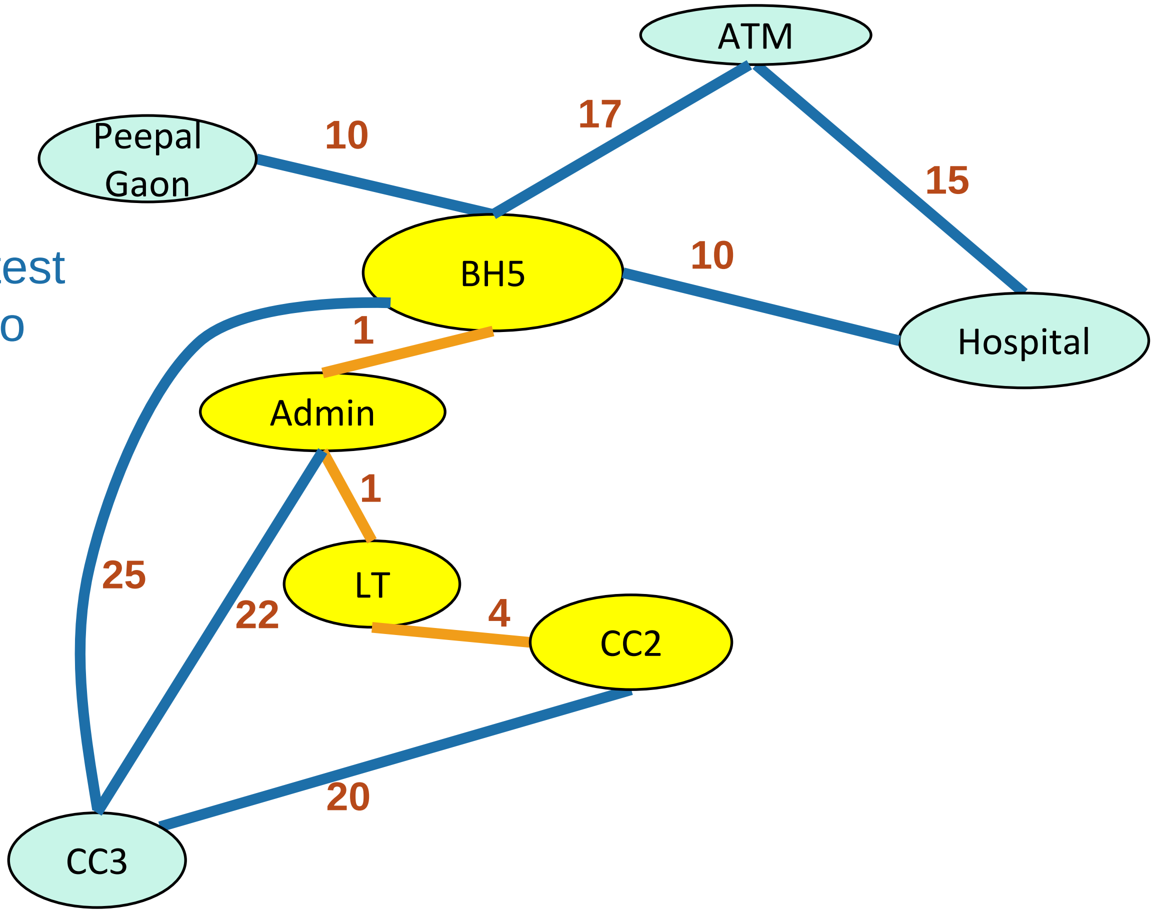


- The **distance** d(u,v) between two vertices u and v is the cost of the shortest path between u and v.

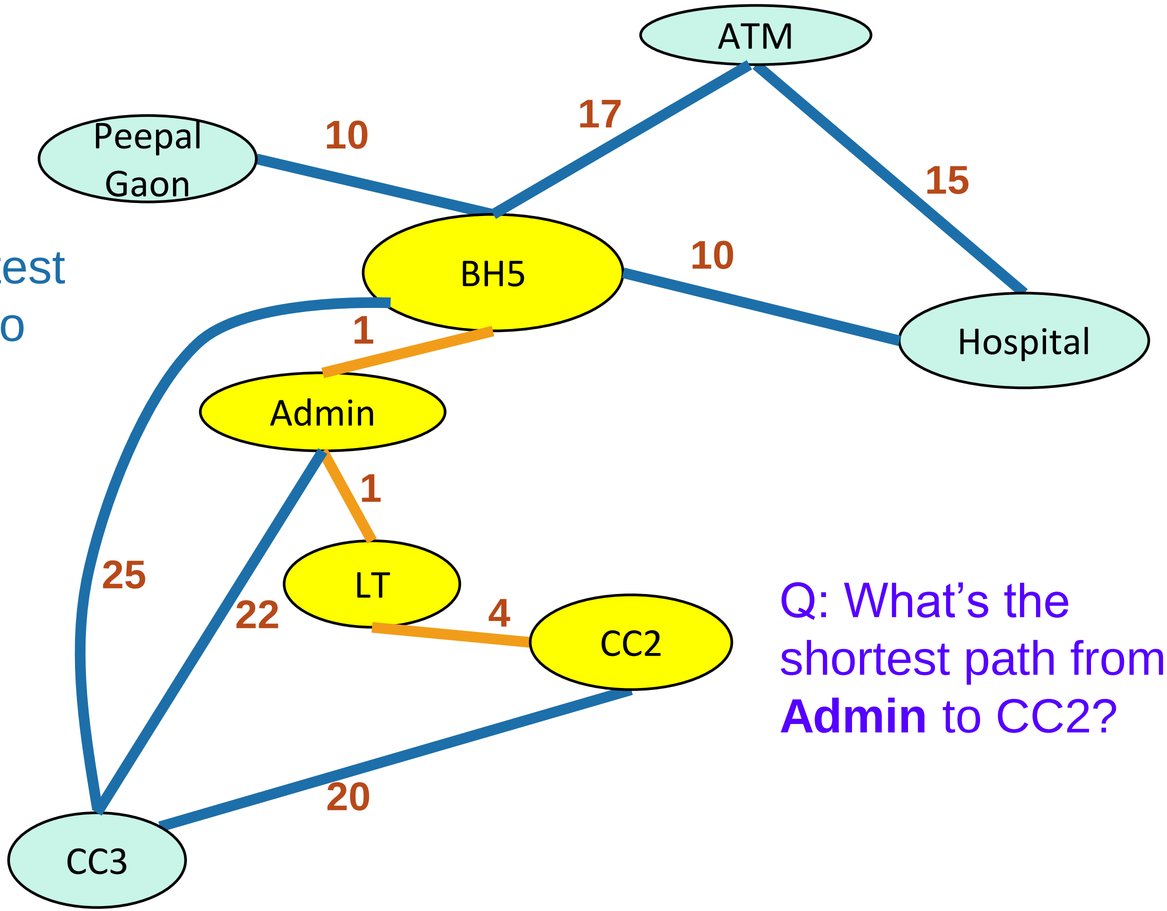
Shortest paths



This is the shortest path from BH5 to CC2.

It has cost 6.

Shortest paths



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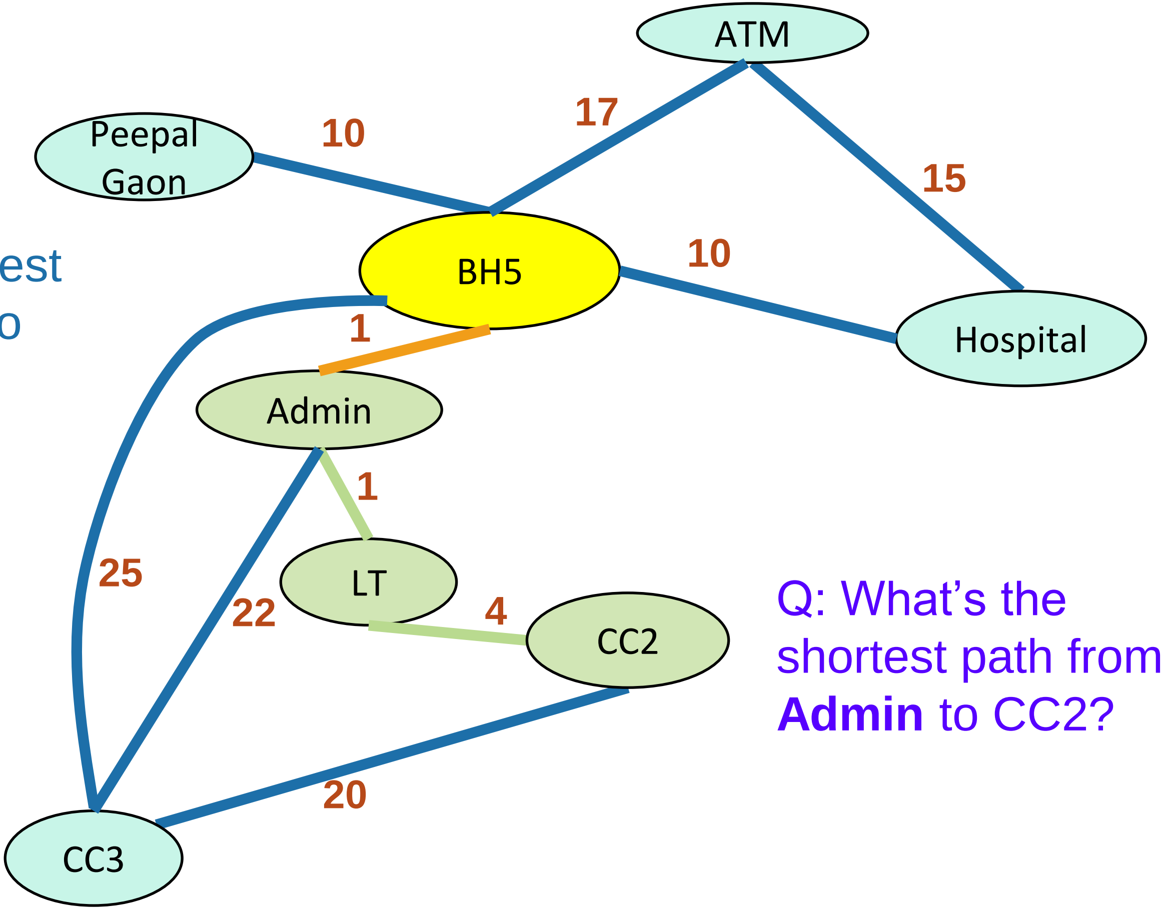
Q: What's the shortest path from **Admin** to CC2?

Shortest paths



This is the shortest path from BH5 to CC2.

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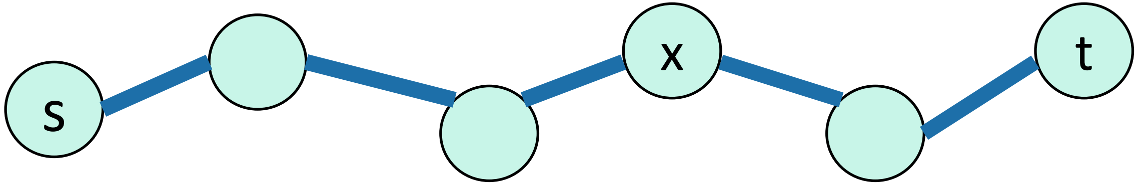
Q: What's the shortest path from **Admin** to CC2?

Warm-up

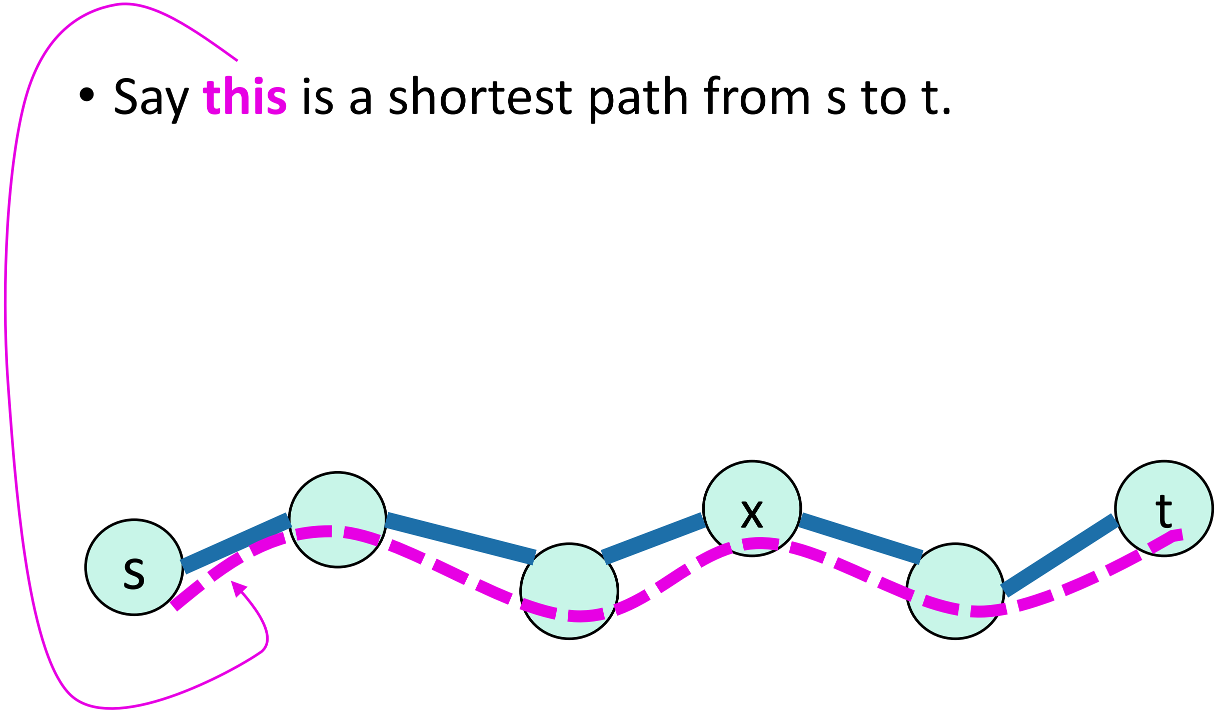
- A sub-path of a shortest path is also a shortest path.



Warm-up

- A sub-path of a shortest path is also a shortest path.

- Say **this** is a shortest path from s to t.



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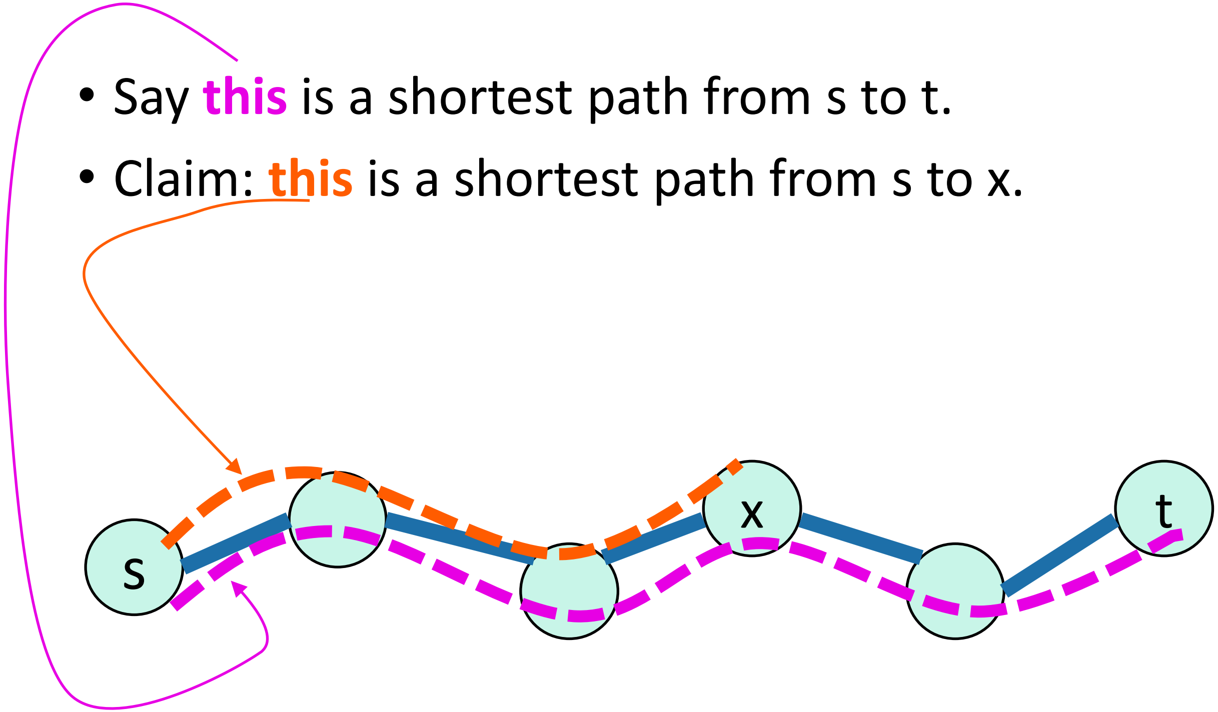
- Claim: **this** is a shortest path from s to x.



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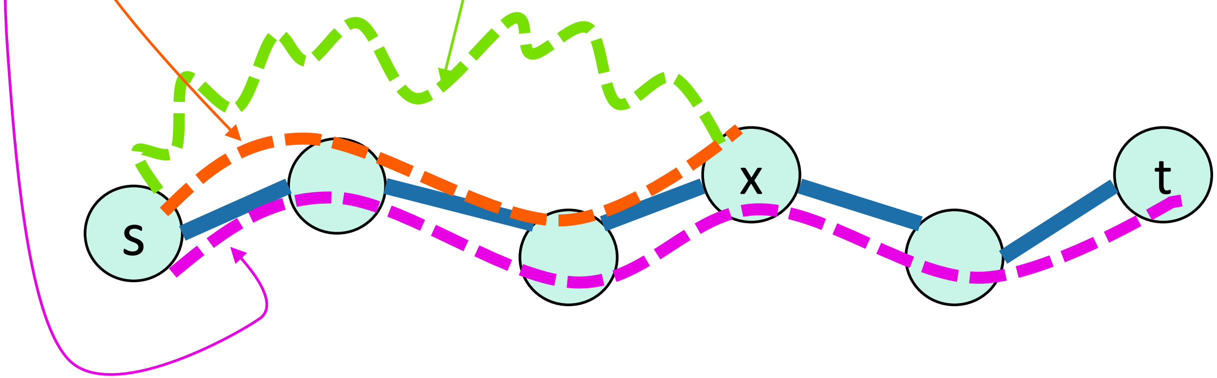

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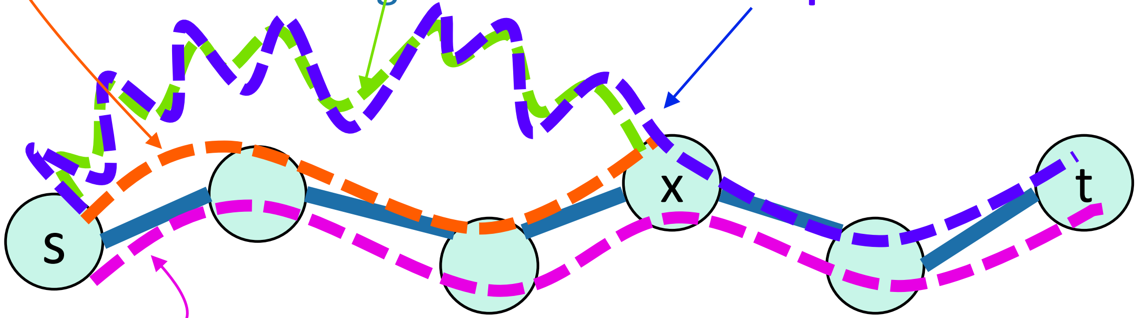
  - But then that gives an **even shorter path** from s to t!



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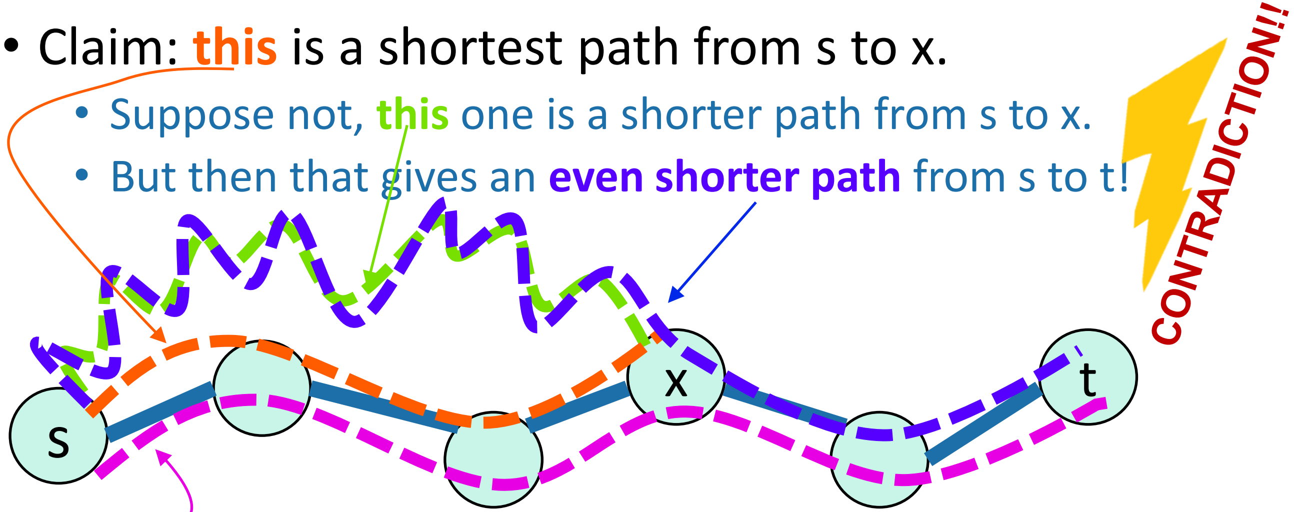

Single-source shortest-path problem

- I want to know the shortest path from one vertex (BH5) to all other vertices.

Single-source shortest-path problem

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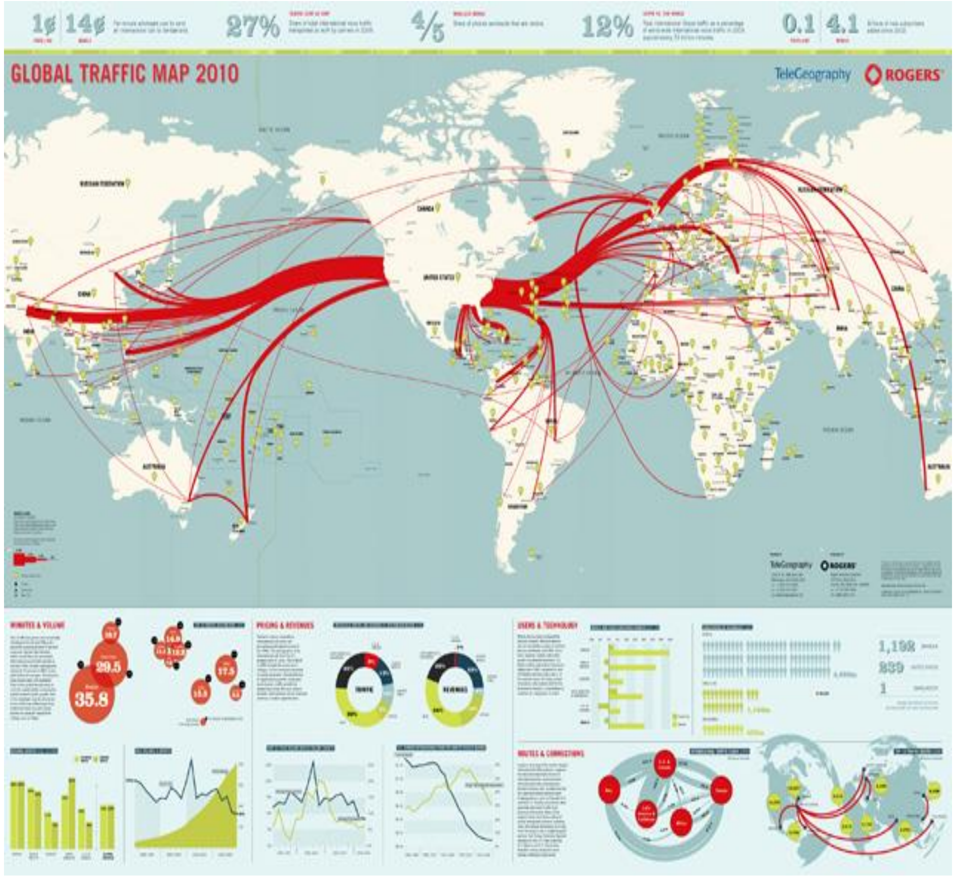
Destination	Cost	To get there
Admin	1	Admin
LT	2	Admin-LT
Peepal Gaon	10	Peepal Gaon
ATM	17	ATM
CC2	6	Admin-LT-CC2
Hospital	10	Hospital
CC3	23	Admin-CC3

Example

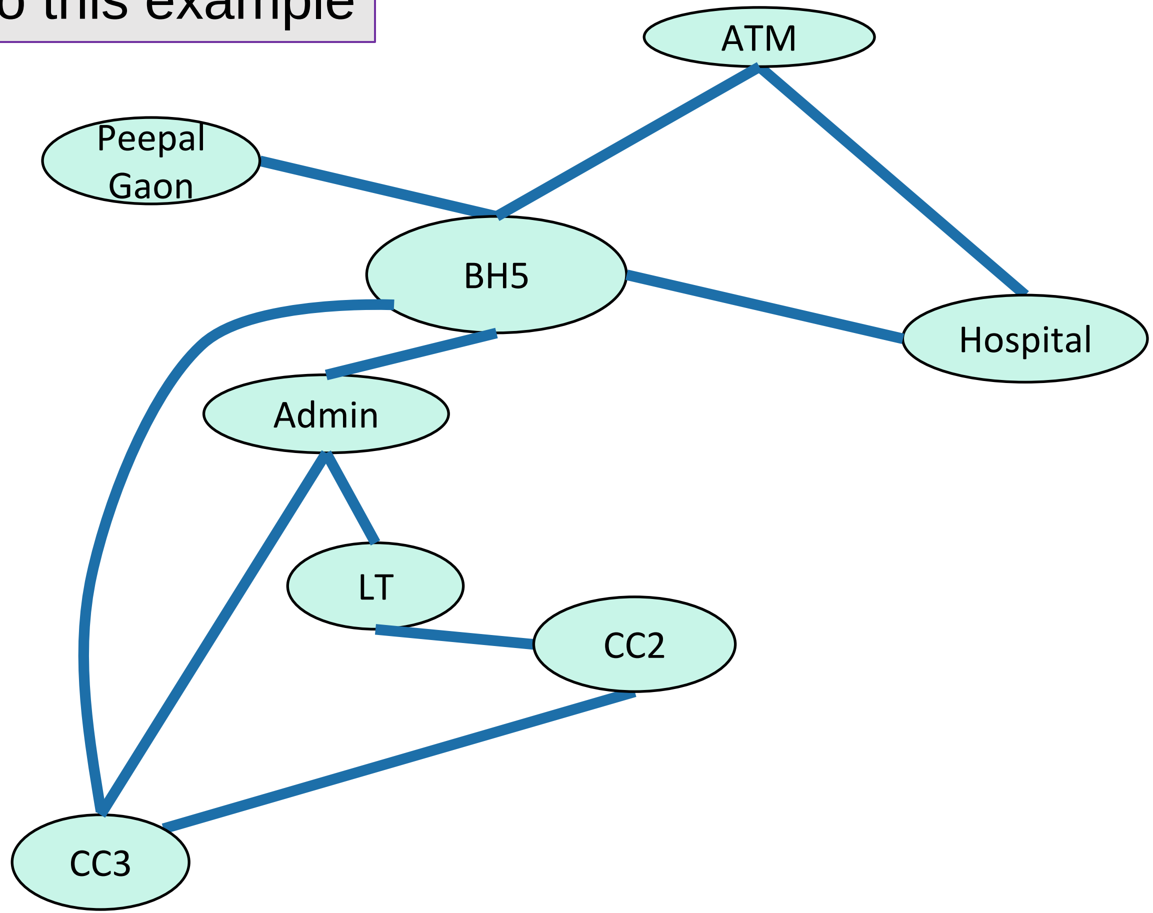
- **“what is the shortest path from IIITA to [anywhere else]”**
- Edge weights have something to do with time, money, hassle.

Example

- **Network routing**
- I send information over the internet, from my computer to all over the world.
- Each path has a cost which depends on link length, traffic, other costs, etc..
- How should we send packets?

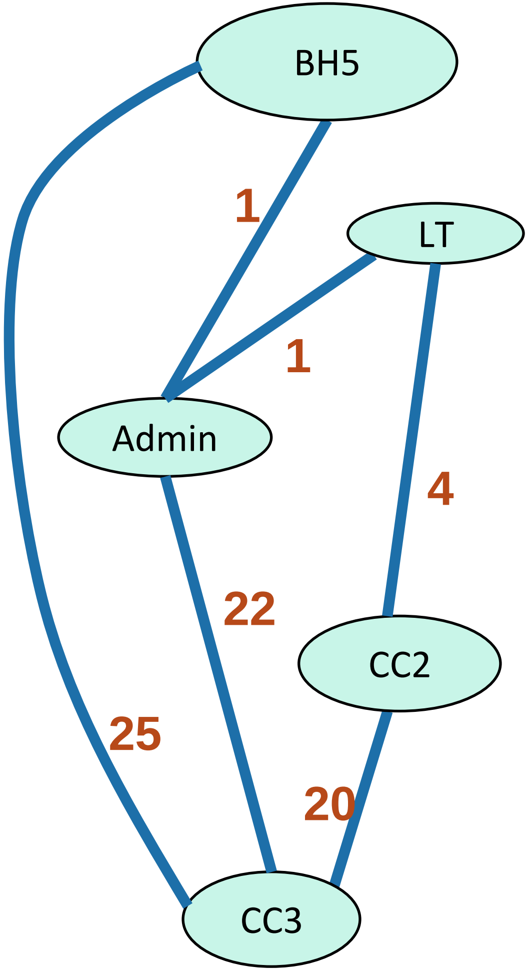


Back to this example



Dijkstra's algorithm

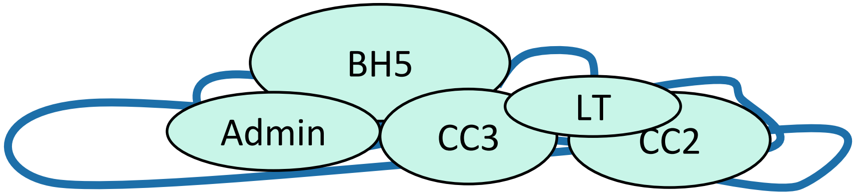
- Finds shortest paths from BH5 to everywhere else.



Dijkstra

intuition

All vertices are on ground initially.



Dijkstra
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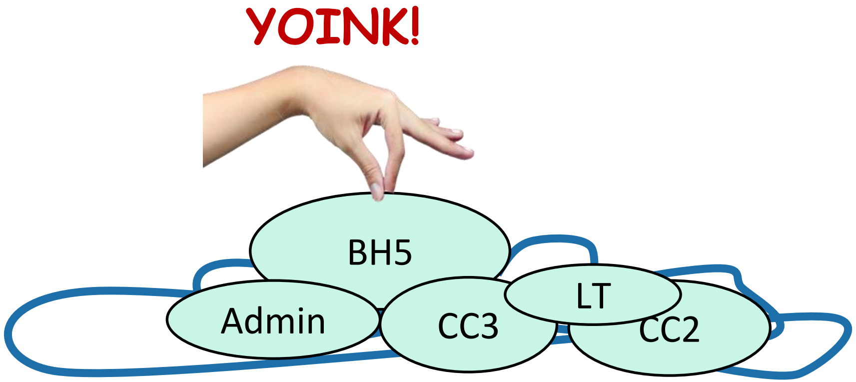
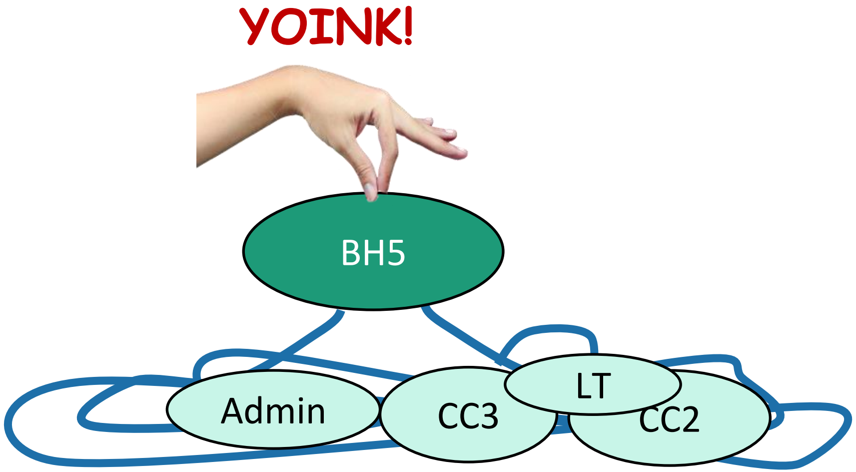


Dijkstra
intuition

A vertex is done when it's not on the ground anymore.



Dijkstra

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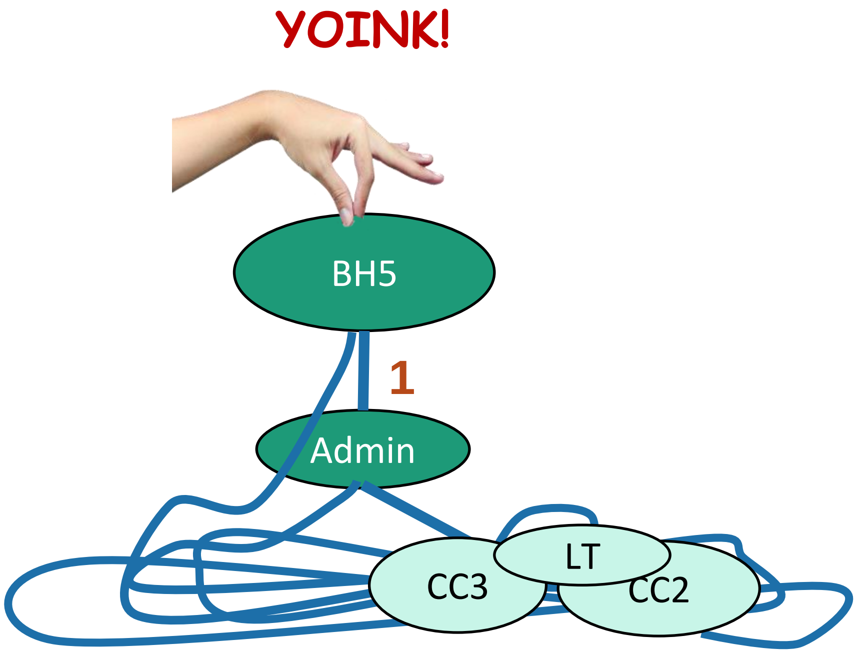

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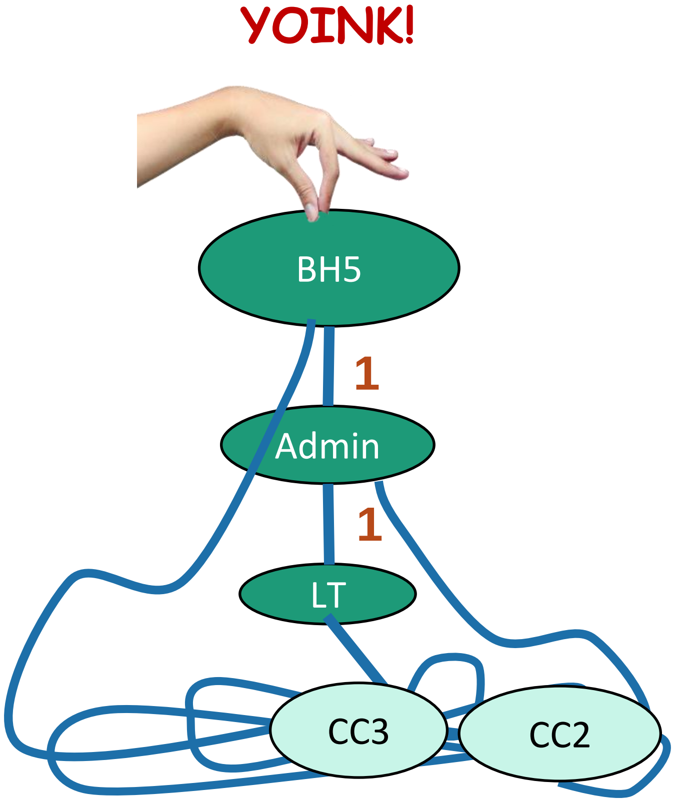

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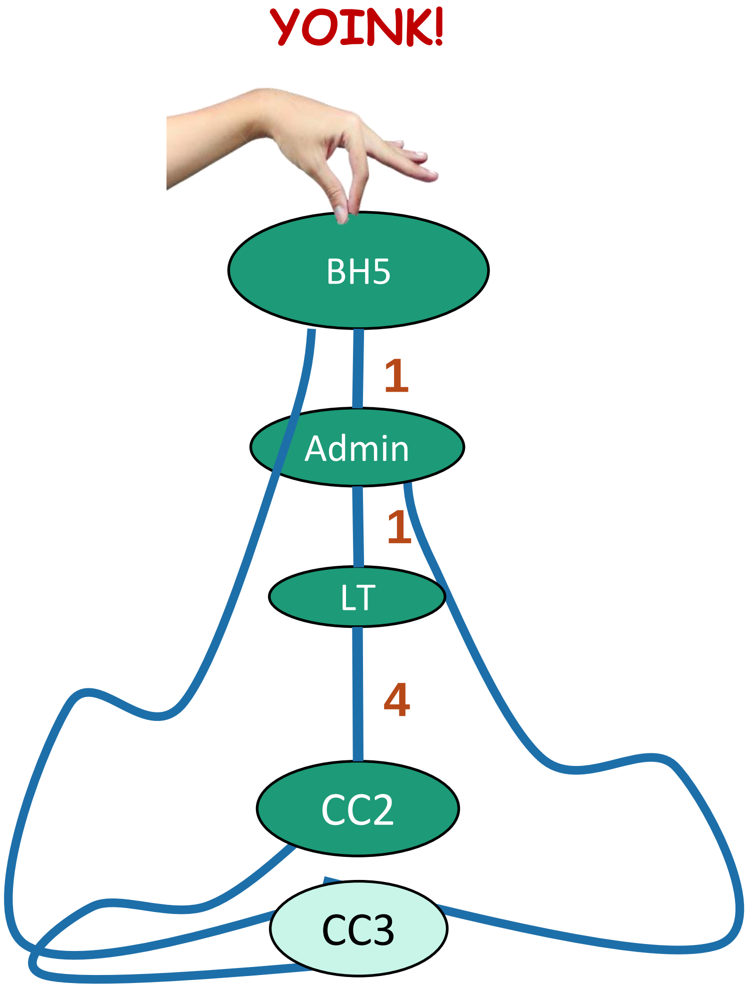

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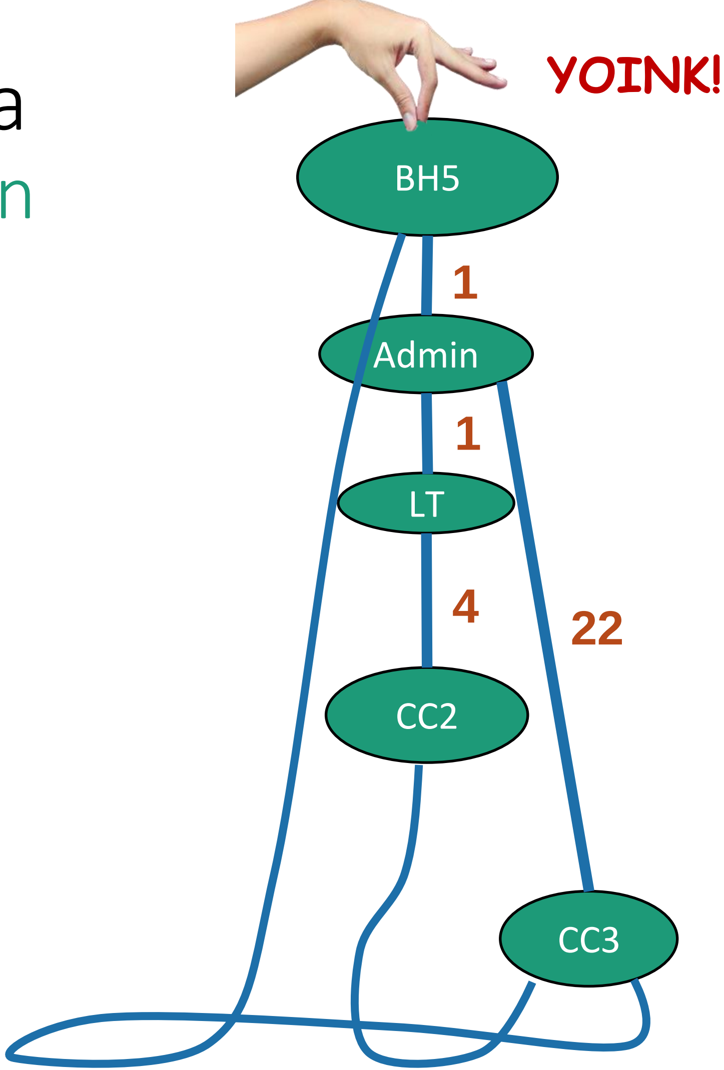
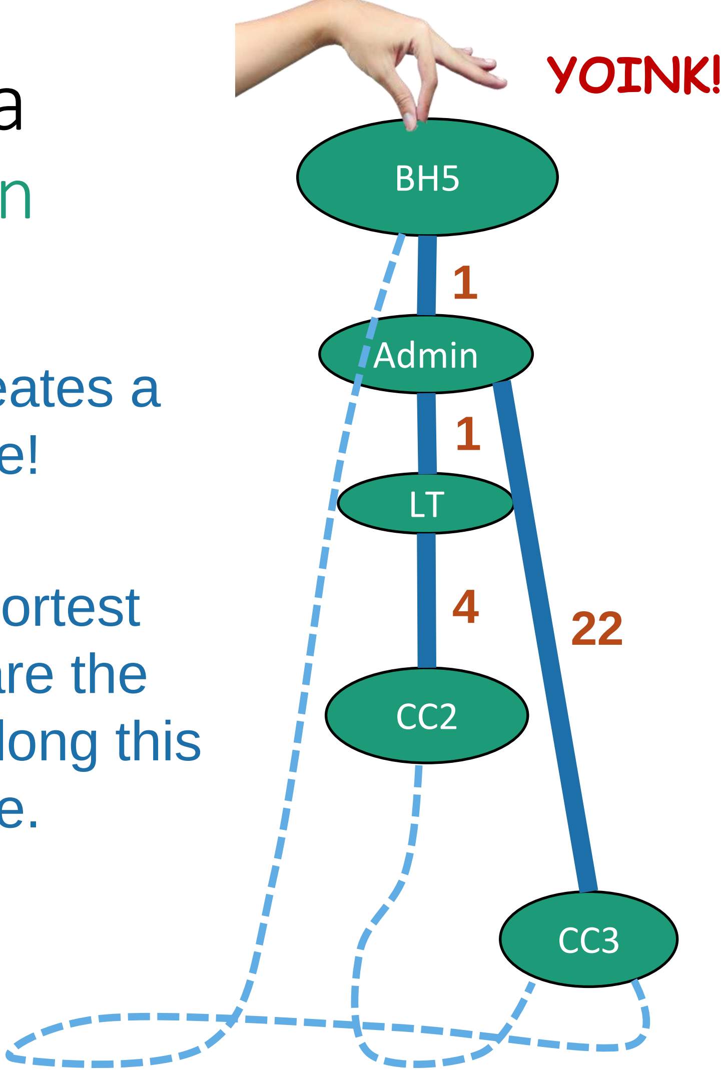


Dijkstra intuition



This creates a tree!

The shortest paths are the lengths along this tree.

How do we actually implement this?

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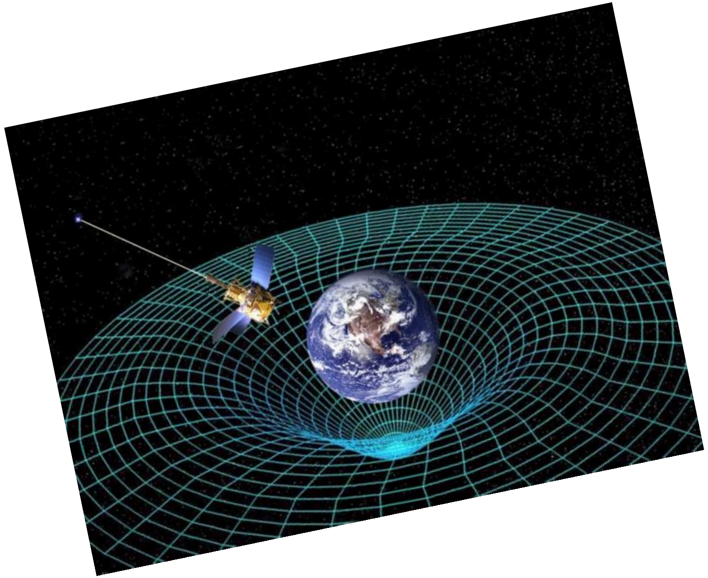
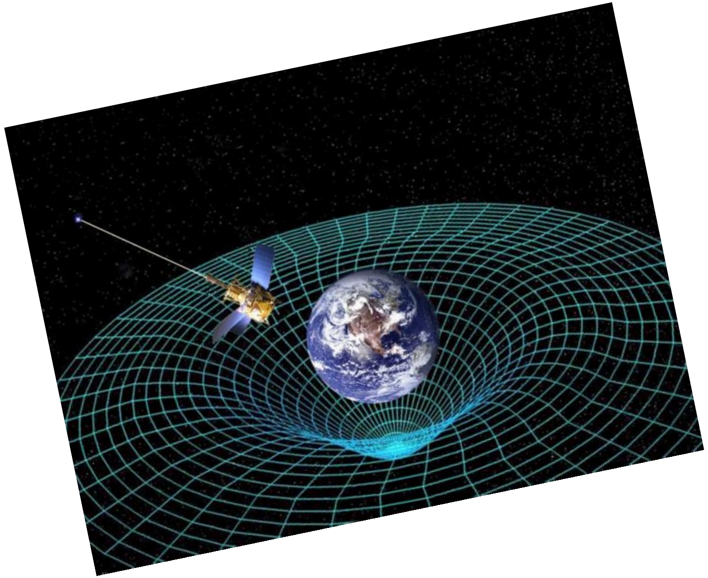


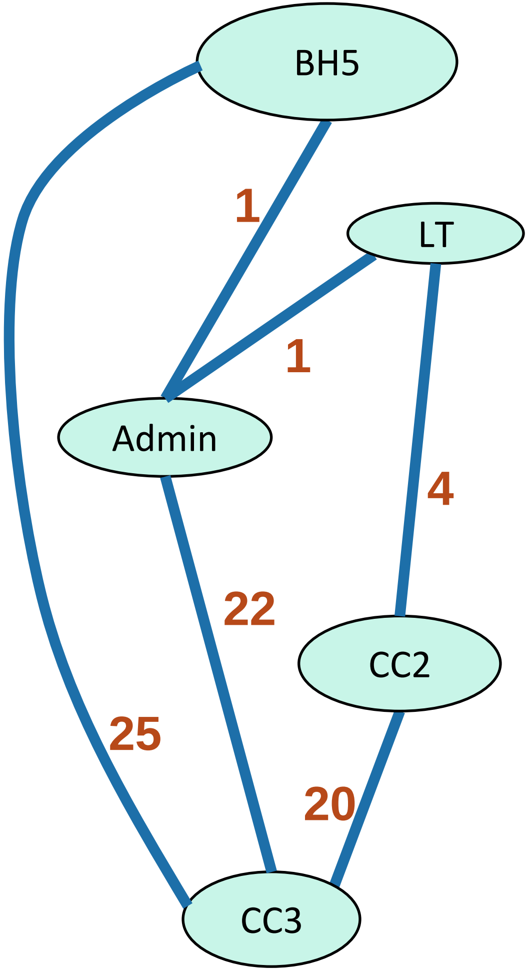
How do we actually implement this?

- **Without** string and gravity?



Dijkstra by example

How far is a node from BH5?



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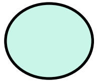
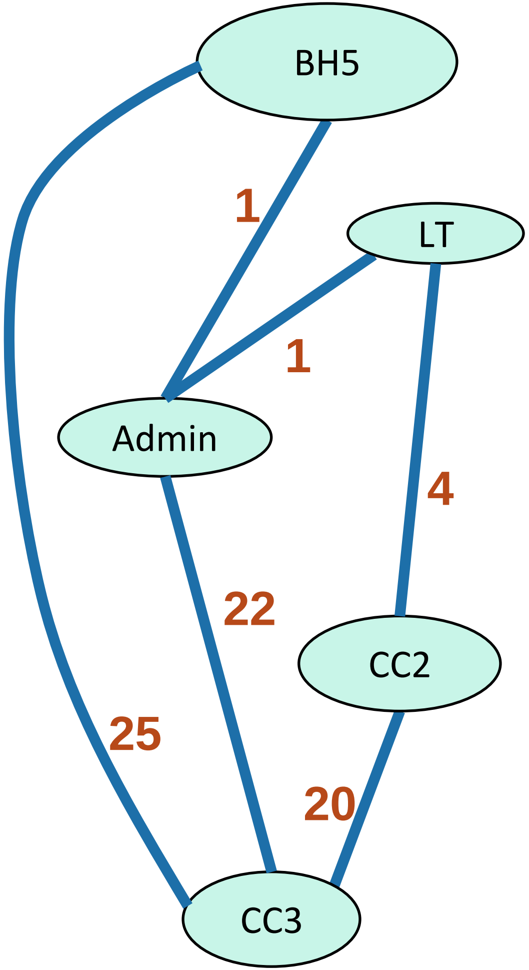


I'm not sure yet



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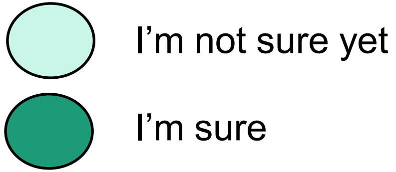

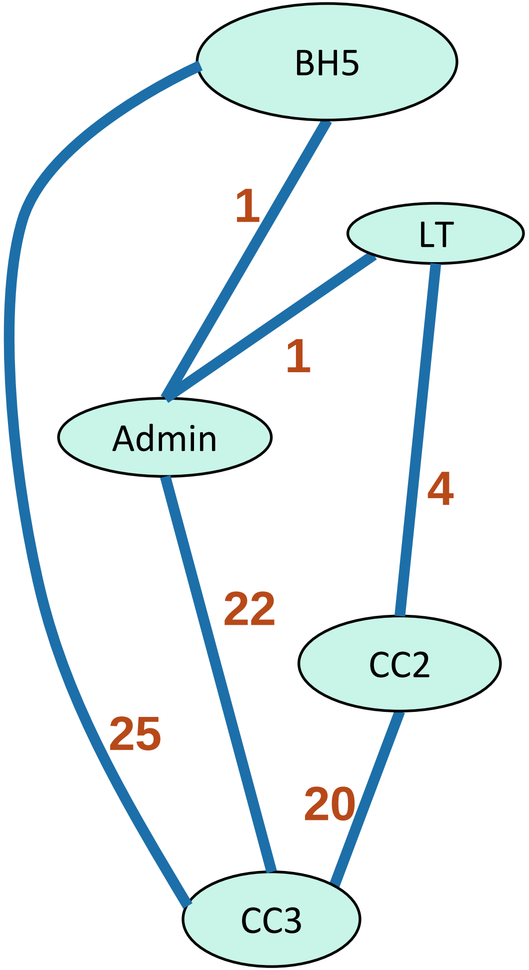

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x **x = d[v]** is my best **over-estimate** for dist(BH5,v).

Initialize d[v] = ∞
for all non-starting vertices v,
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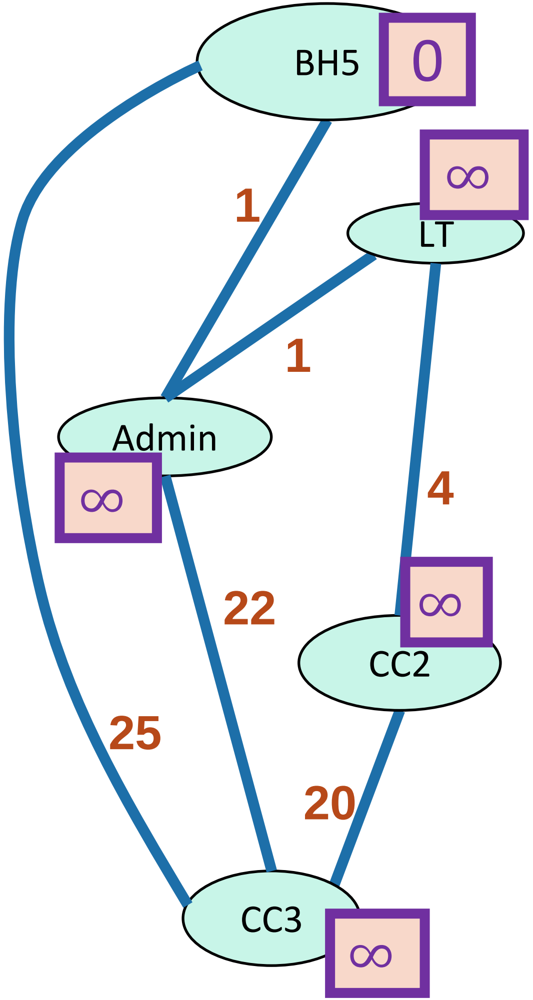

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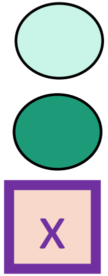

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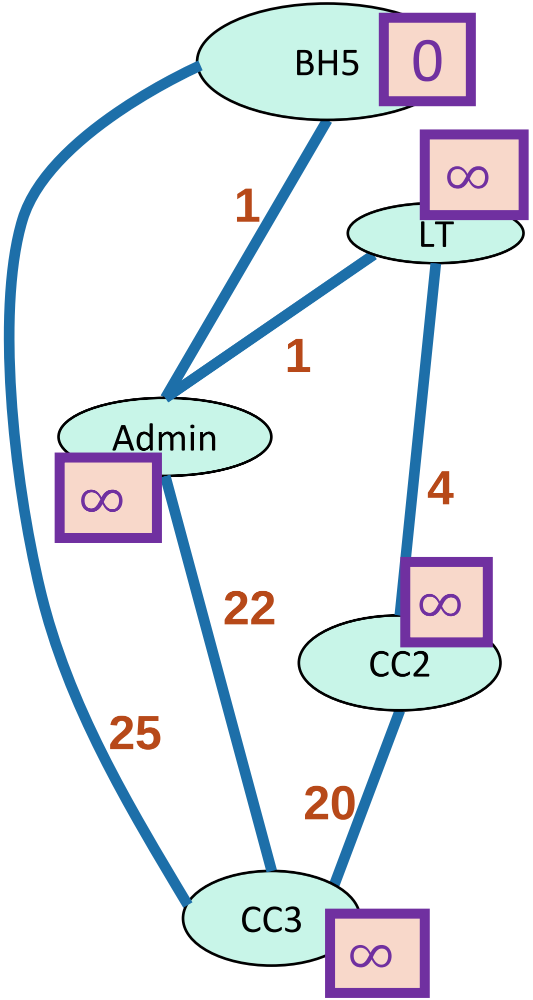

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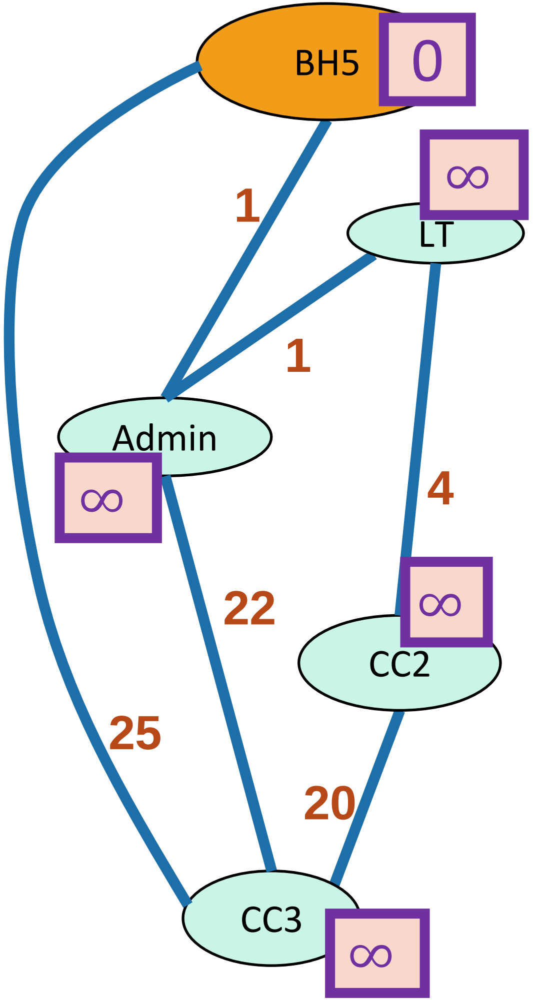

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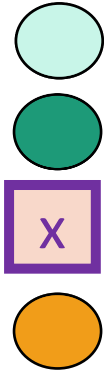

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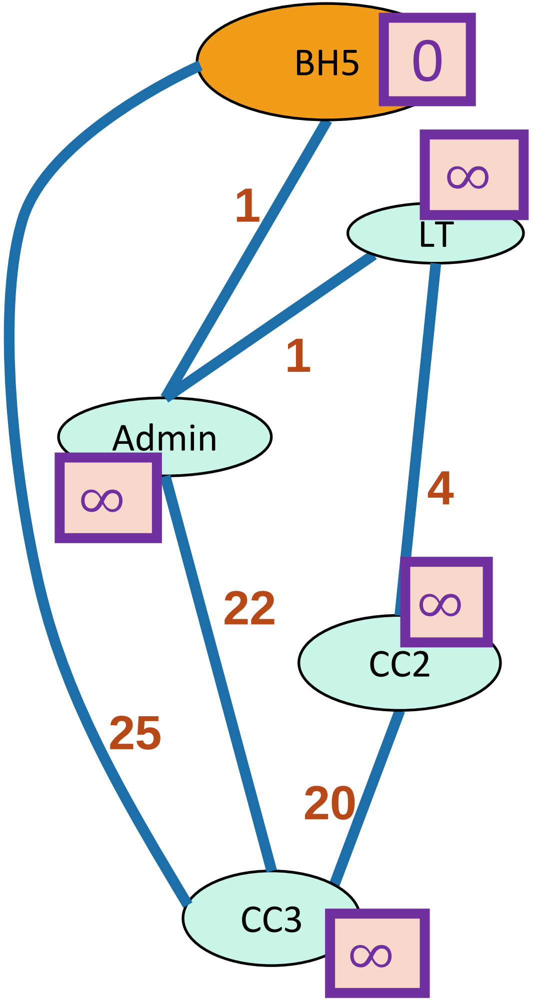

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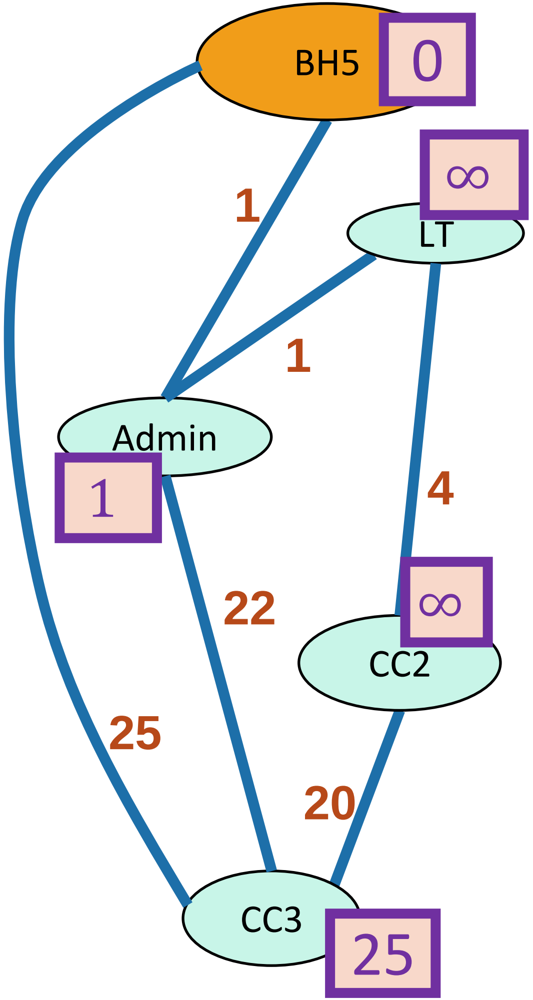

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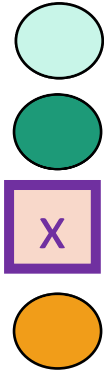

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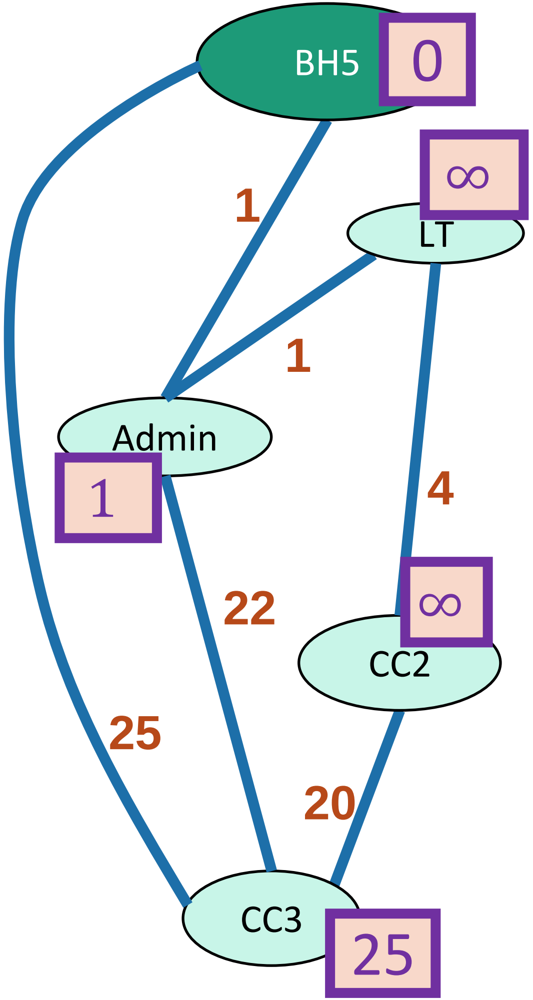

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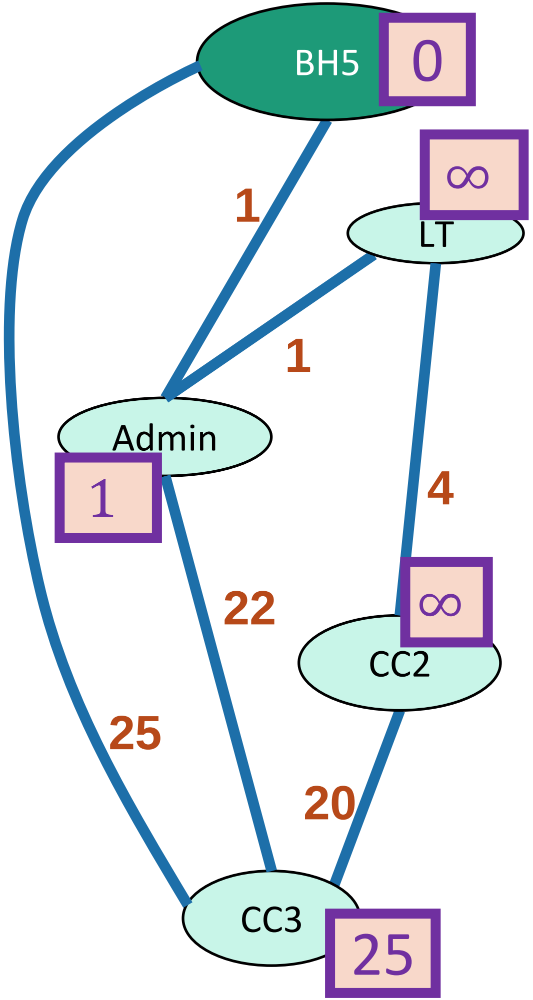

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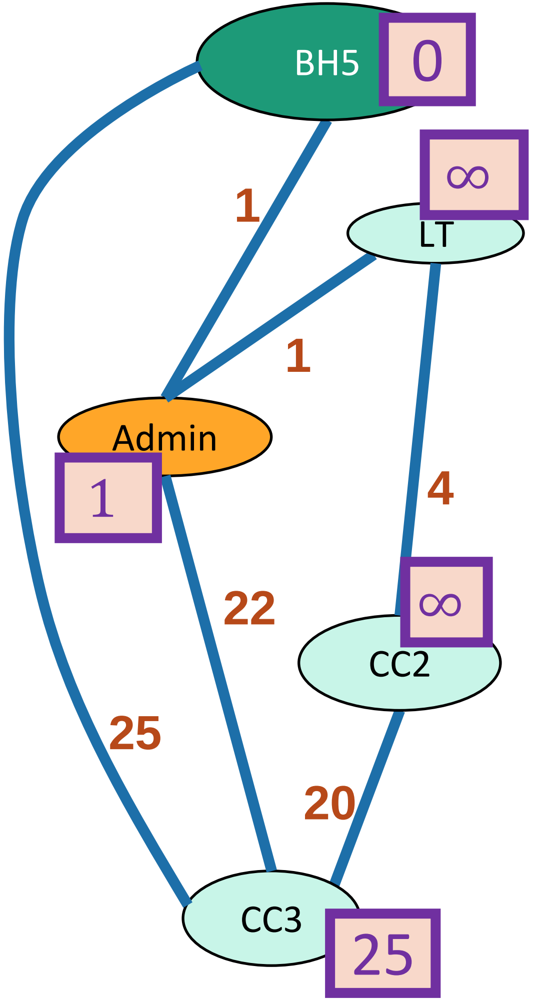

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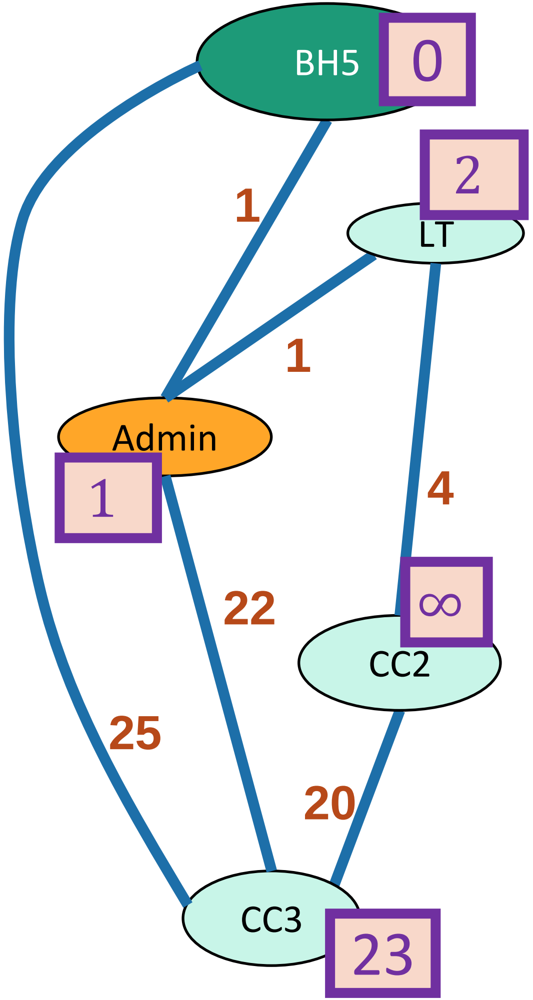

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- Current node u

- Pick the **not-sure** node u with the smallest estimate **d[u].**
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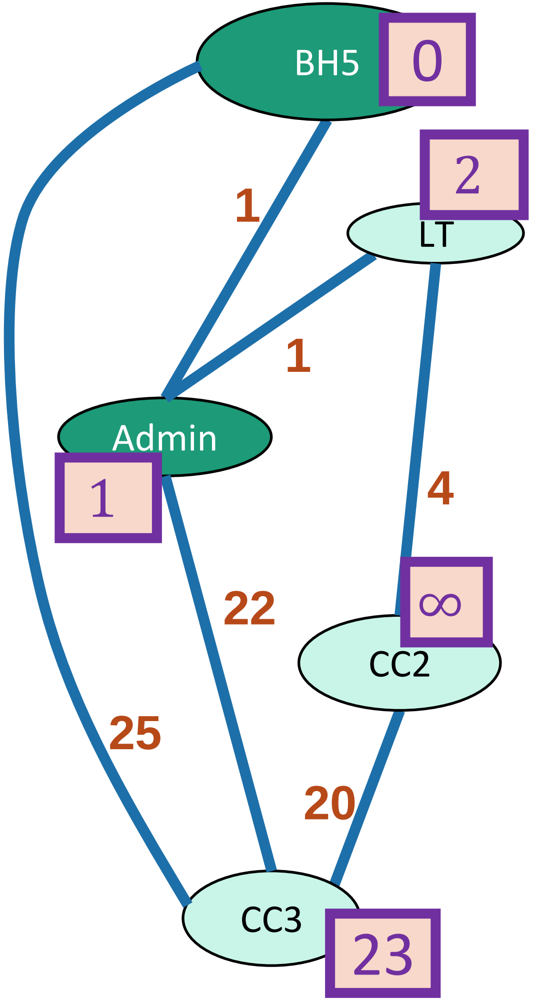
- Repeat



Dijkstra by example

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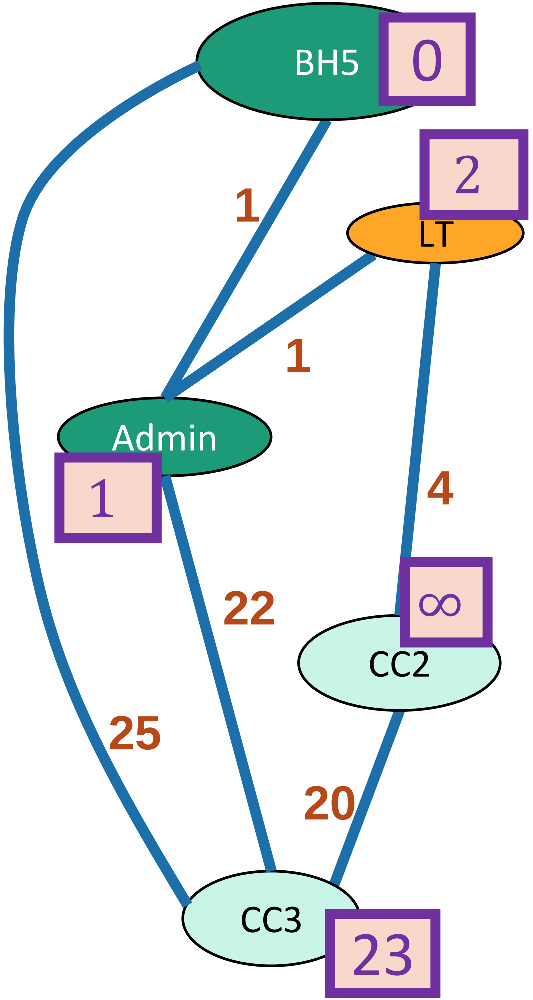

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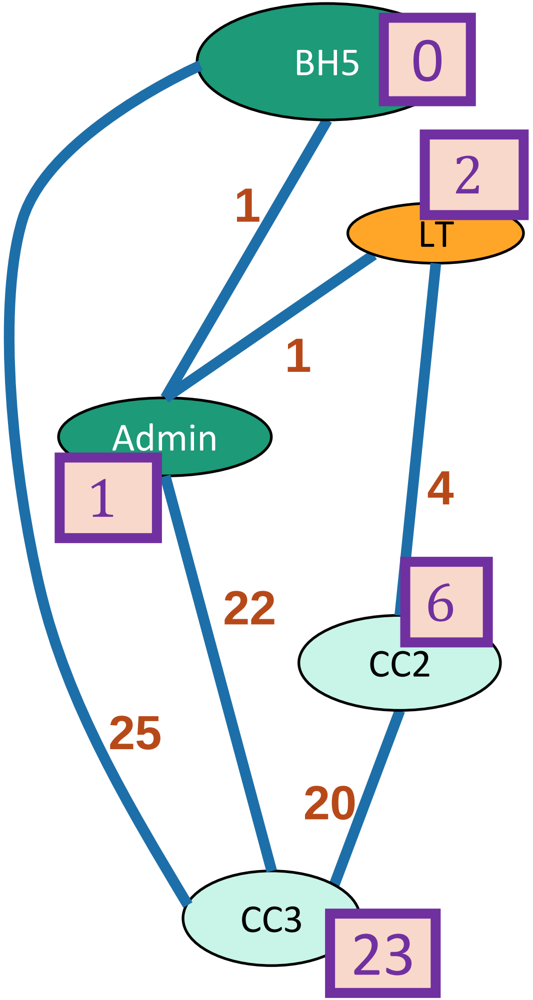

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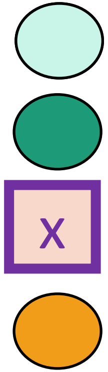

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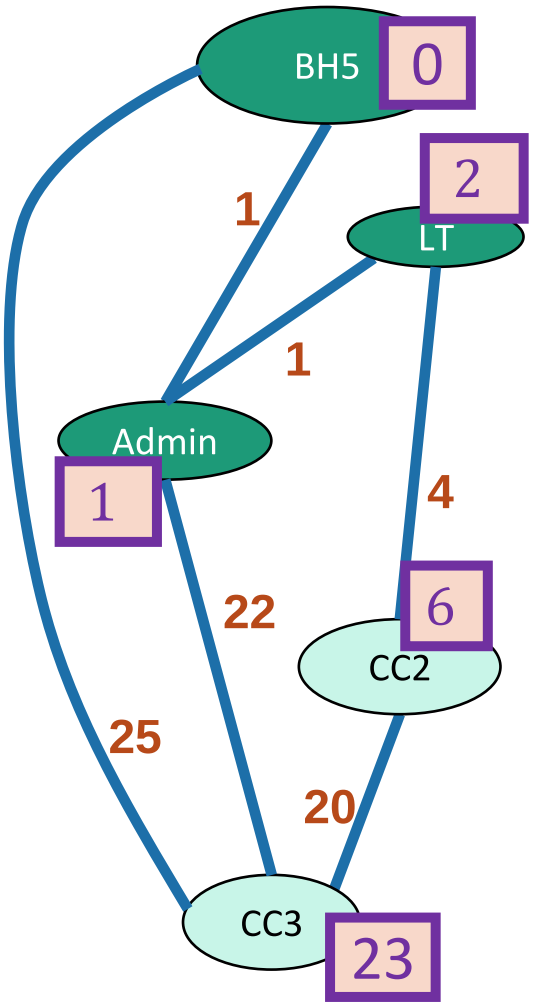

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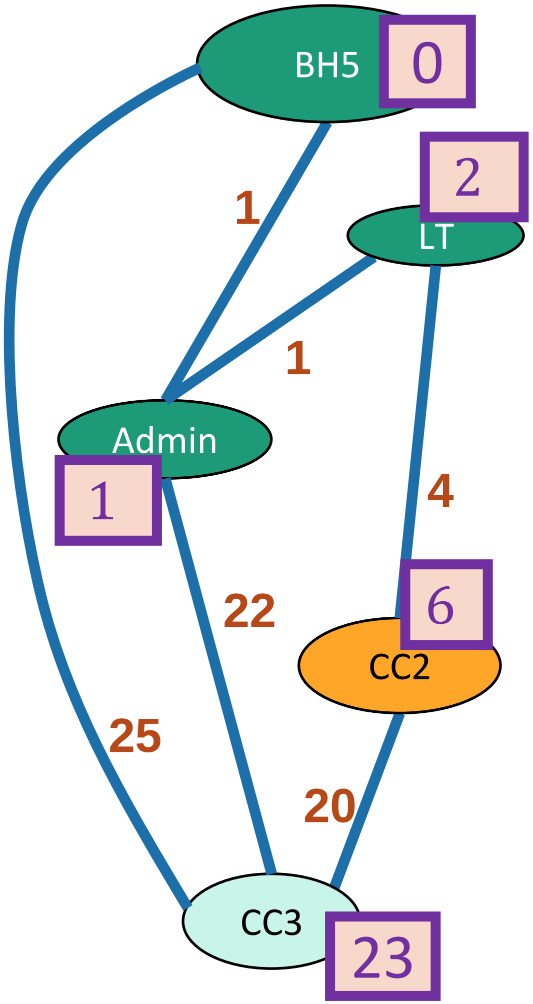

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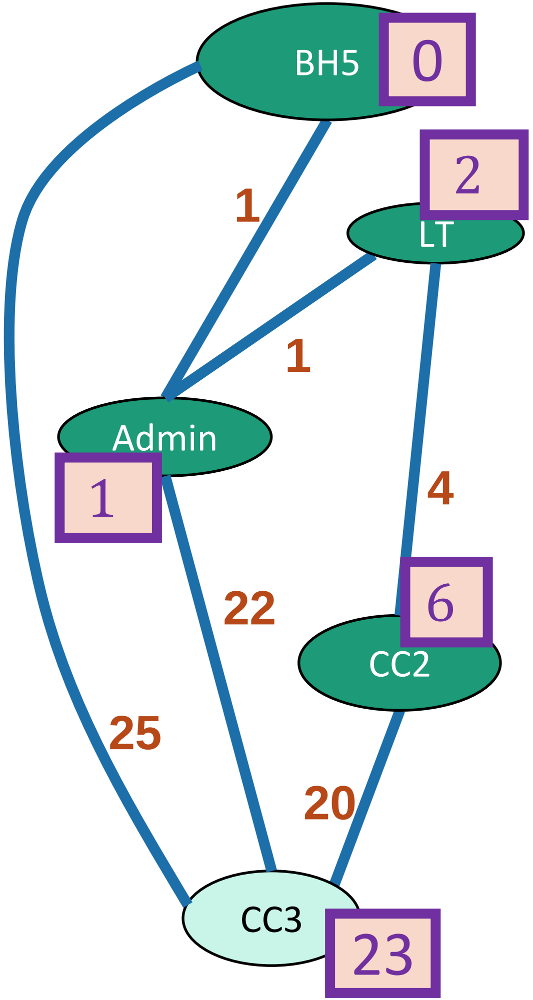

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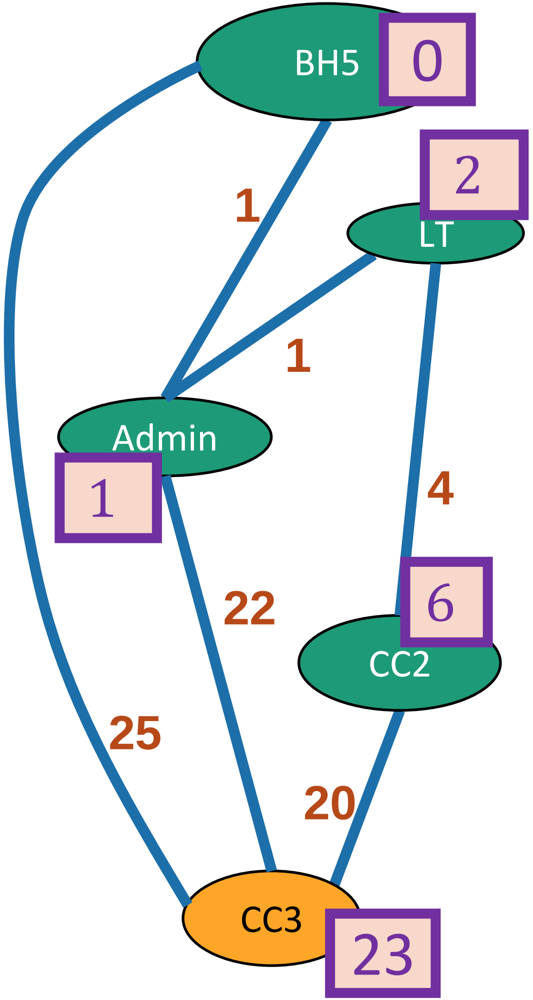

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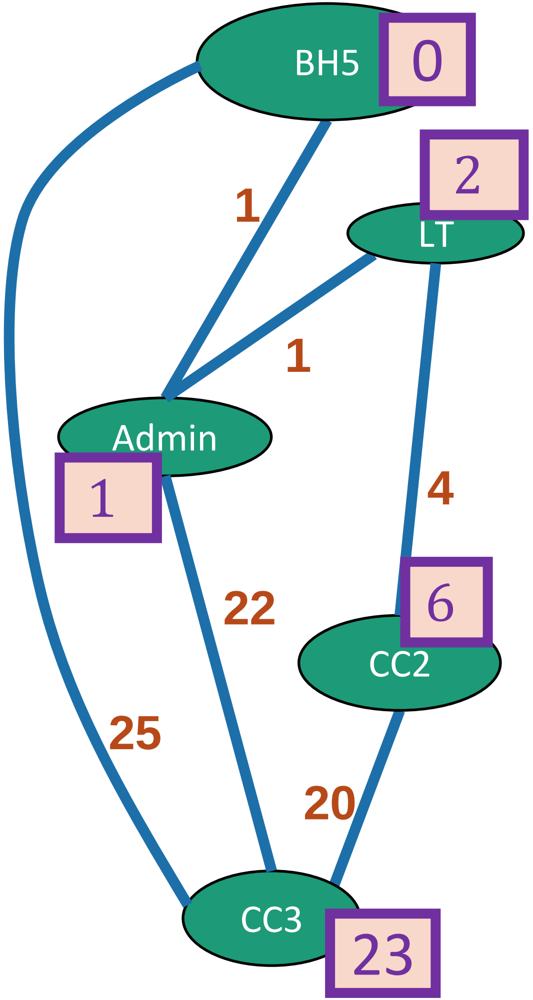

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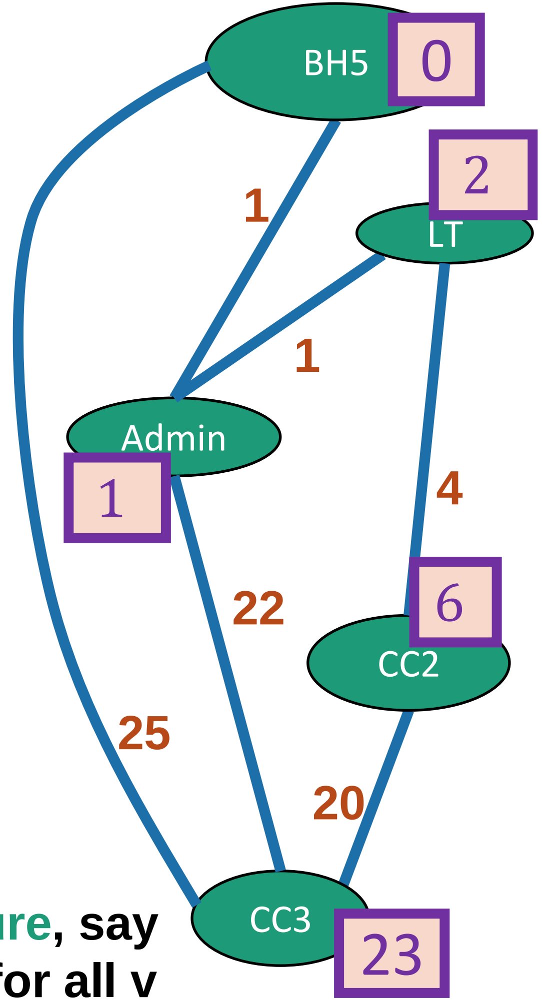

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- Mark u as **sure**.
- Repeat

- **After all nodes are sure, say that d(BH5, v) = d[v] for all v**

Dijkstra’s algorithm

Dijkstra(G,s):

- Set all vertices to **not-sure**
- $d[v] = \infty$ for all v in V
- $d[s] = 0$
- **While** there are **not-sure** nodes:
  - Pick the **not-sure** node u with the smallest estimate **d[u].**
  - **For** v in u.neighbors:
    - $d[v] \leftarrow \min(d[v] , d[u] + \text{edgeWeight}(u,v))$
  - Mark u as **sure**.
- Now $d(s, v) = d[v]$

Lots of implementation details left un-explained.
We’ll get to that!

As usual

- Does it work?

- Is it fast?

As usual

- Does it work?
  - Yes.

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  - Depends on how you implement it.

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Why does this work?

- **Theorem**:
  - Suppose we run Dijkstra on G =(V,E), starting from s.
  - At the end of the algorithm, the estimate **d[v]** is the actual distance d(s,v).

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  - **Claim 1**: For all v, **d[v] $\geq$ d(s,v).**
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- **Claims 1 and 2** imply the **theorem.**
  - When v is marked **sure**, **d[v] = d(s,v).** ← Claim 2
  - **d[v] $\geq$ d(s,v)** and never increases, so after v is **sure**, **d[v]** stops changing. ← Claim 1 + def of algorithm
  - This implies that at any time *after* v is marked **sure**, **d[v] = d(s,v).**
  - All vertices are **sure** at the end, so all vertices end up with **d[v] = d(s,v).**

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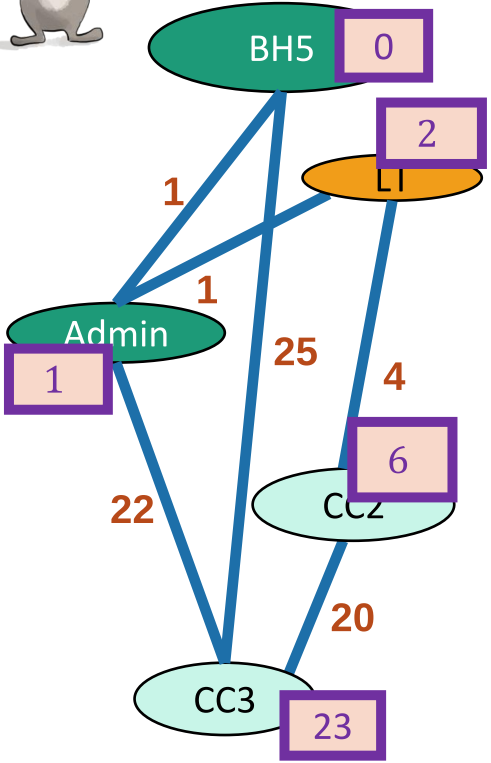
Next let's prove the claims!

Claim 1

d[v] ≥ d(s,v) for all v.



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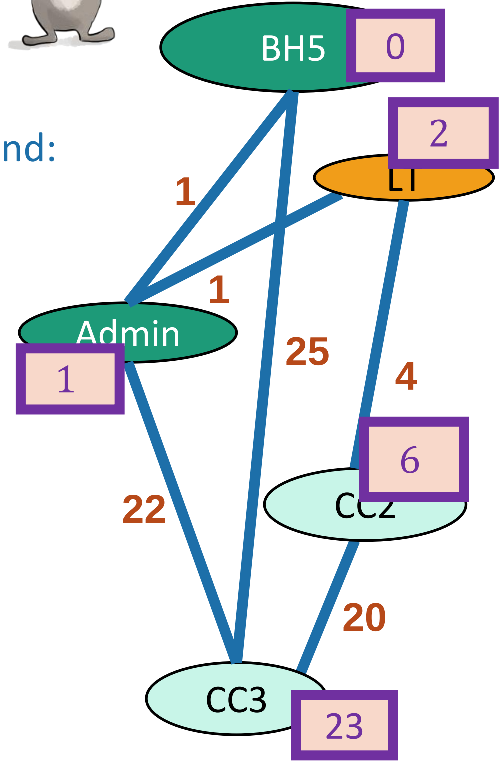
$d[v] \geq d(s,v)$ for all v.

Informally:

- Every time we update d[v], we have a path in mind:



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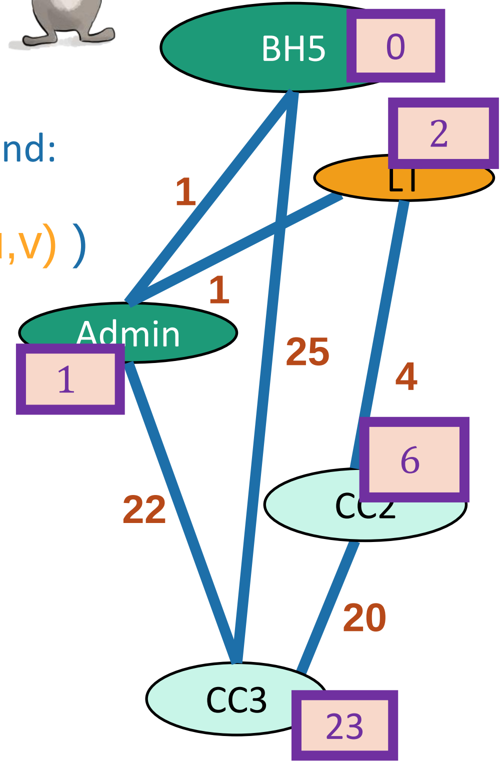

Intuition!

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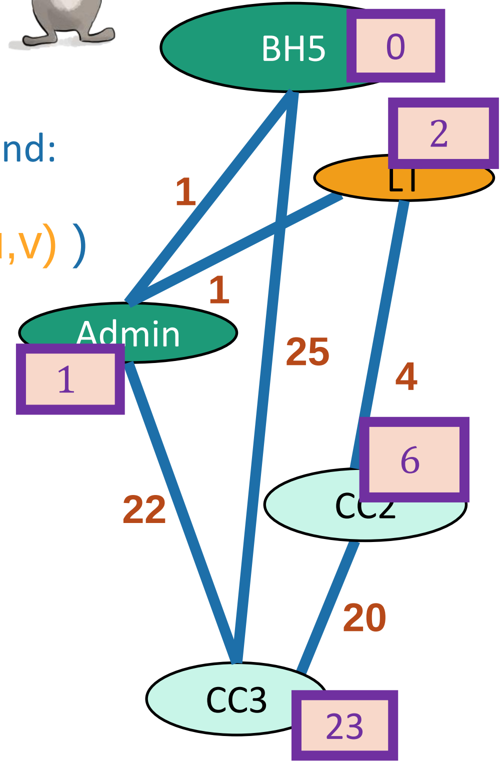
Whatever path we had in mind before

The shortest path to u, and then the edge from u to v.



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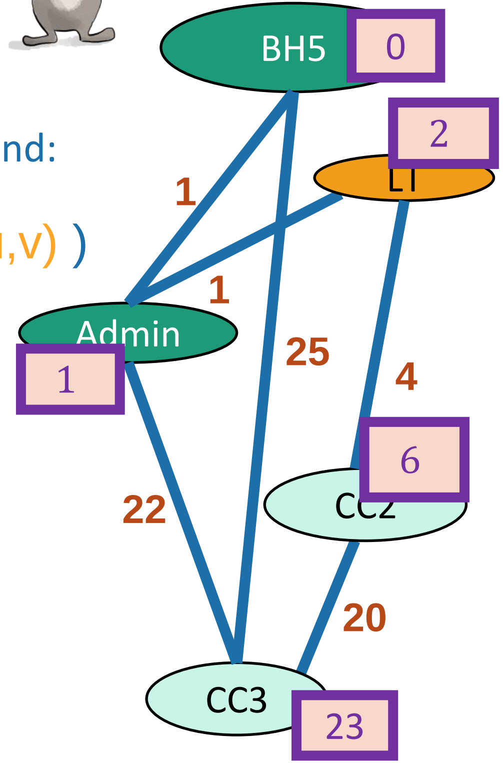
- d[v] = length of the path we have in mind
  $\geq$ length of shortest path
  = d(s,v)



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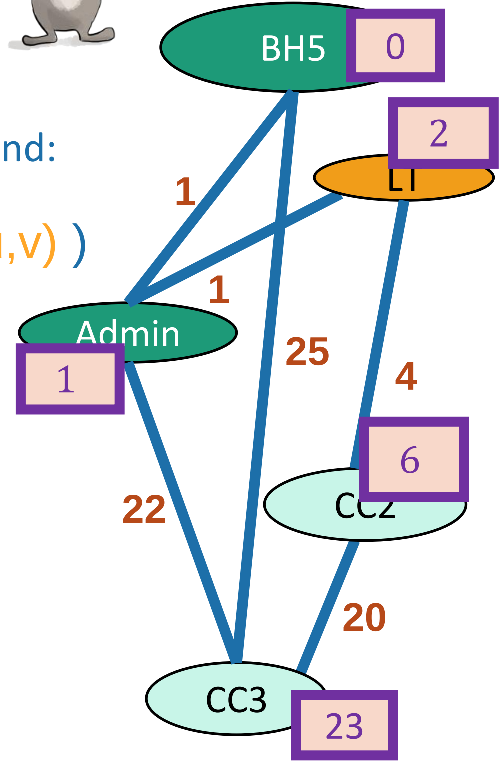


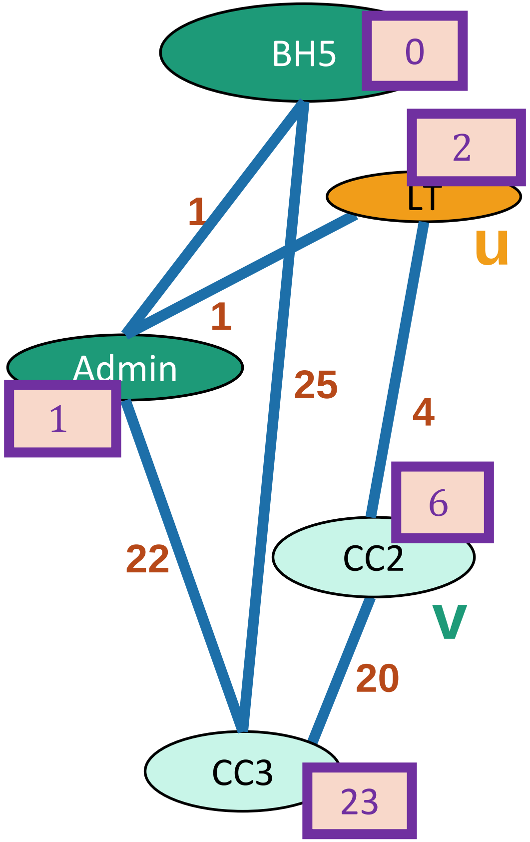
Formally:

- We should prove this by induction.

Claim 1

d[v] ≥ d(s,v) for all v.

- Inductive hypothesis.
  - After t iterations of Dijkstra, d[v] ≥ d(s,v) for all v.



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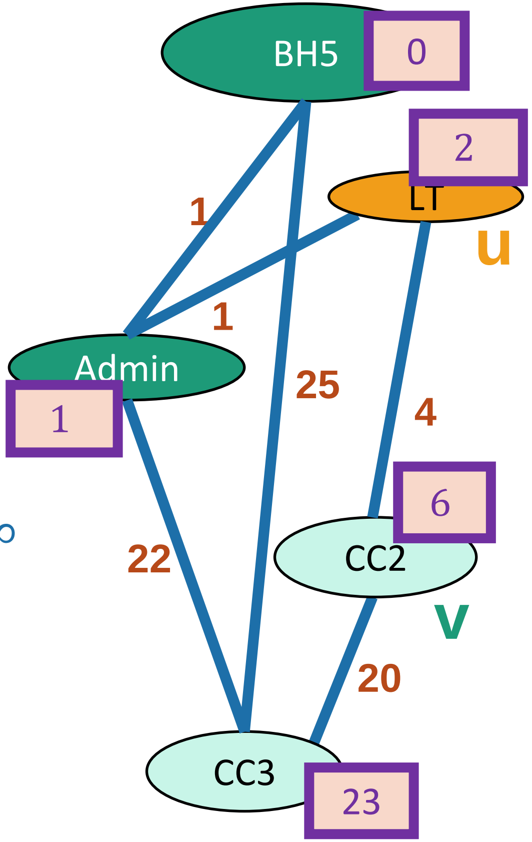

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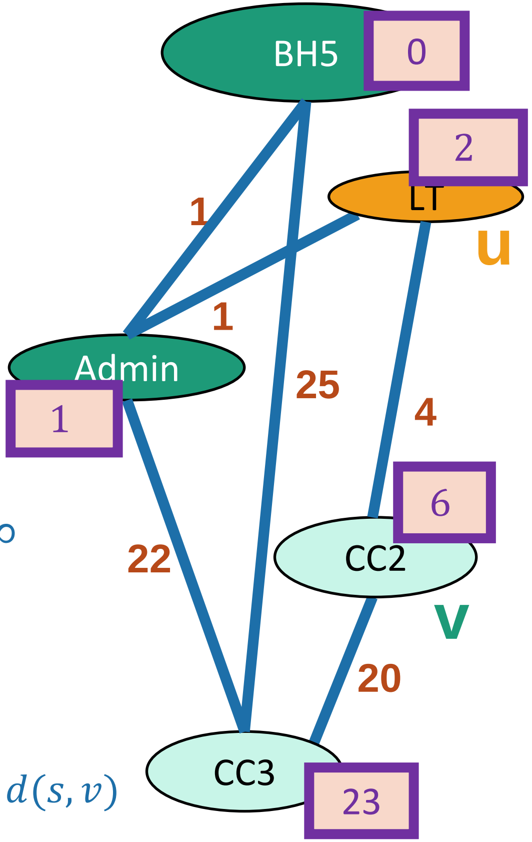
  - At step 0, $d(s,s) = 0$, and $d(s,v) \leq \infty$
- Inductive step: say hypothesis holds for t.
  - At step t+1:
    - Pick **u**; for each neighbor **v**:
    - $d[v] \leftarrow \min(\, d[v]\, ,\, d[u] + w(u,v)\,) \geq d(s,v)$

By induction,
$d[v] \geq d(s,v)$

$d[v] = d[u] + w(u,v)$
$\geq d(s,u) + w(u,v) \geq d(s,v)$
using induction again for d[u]



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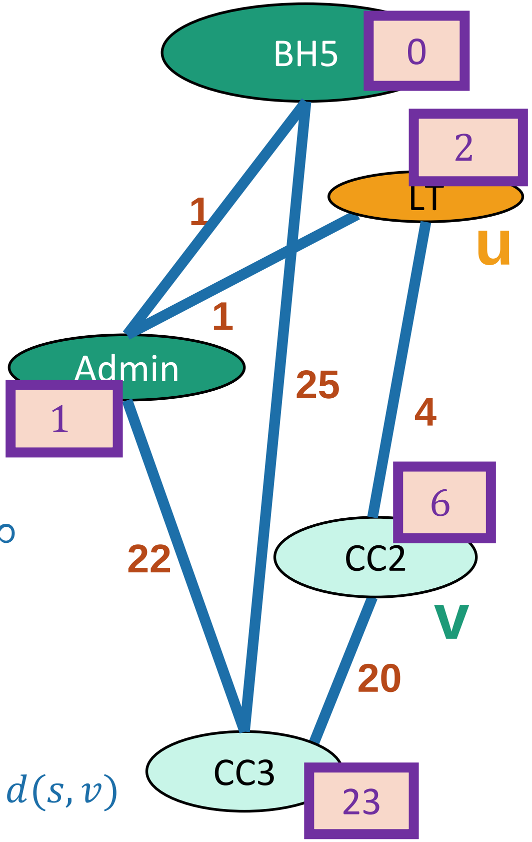

So the inductive hypothesis holds for t+1, and Claim 1 follows.

By induction, $d[v] \geq d(s,v)$

$d[v] = d[u] + w(u,v)$
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Claim 2

When a vertex u is marked sure, $d[u] = d(s,u)$

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- Inductive step:
  - Suppose that we are about to add u to the **sure** list.
  - That is, we picked u in the first line here:

> - Pick the **not-sure** node u with the smallest estimate **d[u].**
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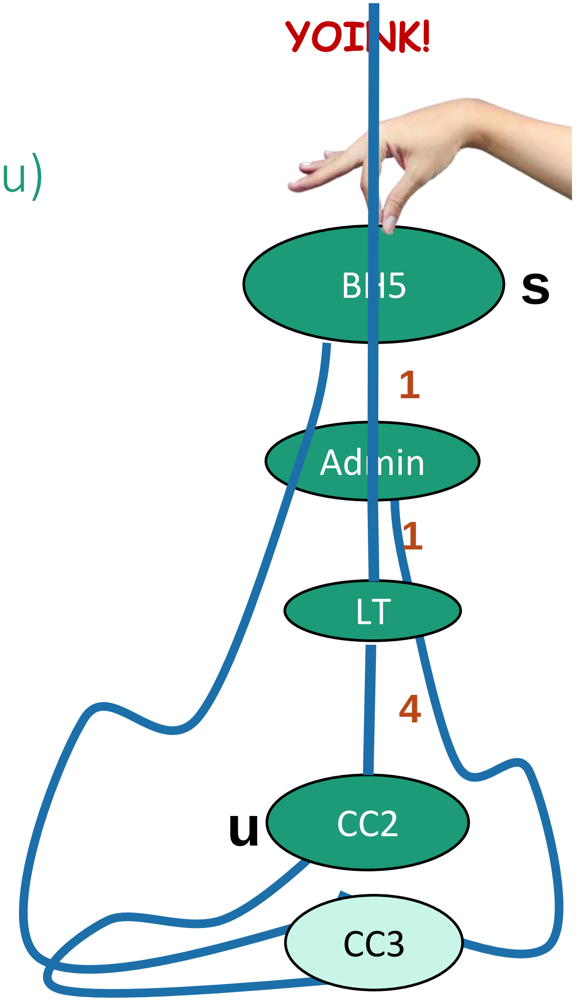
  - Assume by induction that every v already marked **sure** has d[v] = d(s,v).
  - Want to show that d[u] = d(s,u).

Intuition

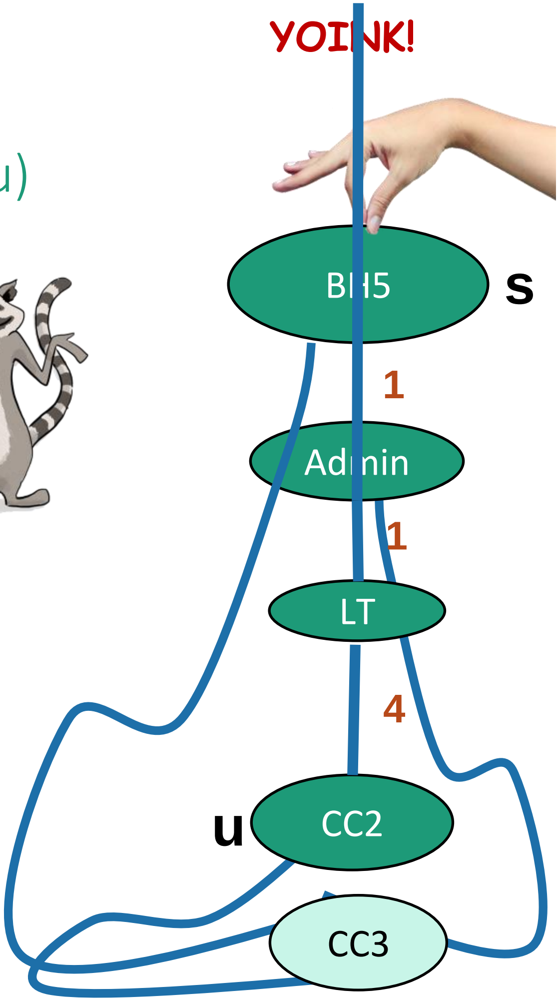
When a vertex u is marked sure, d[u] = d(s,u)



Intuition

When a vertex u is marked sure, d[u] = d(s,u)

- The first path that lifts **u** off the ground is the shortest one.



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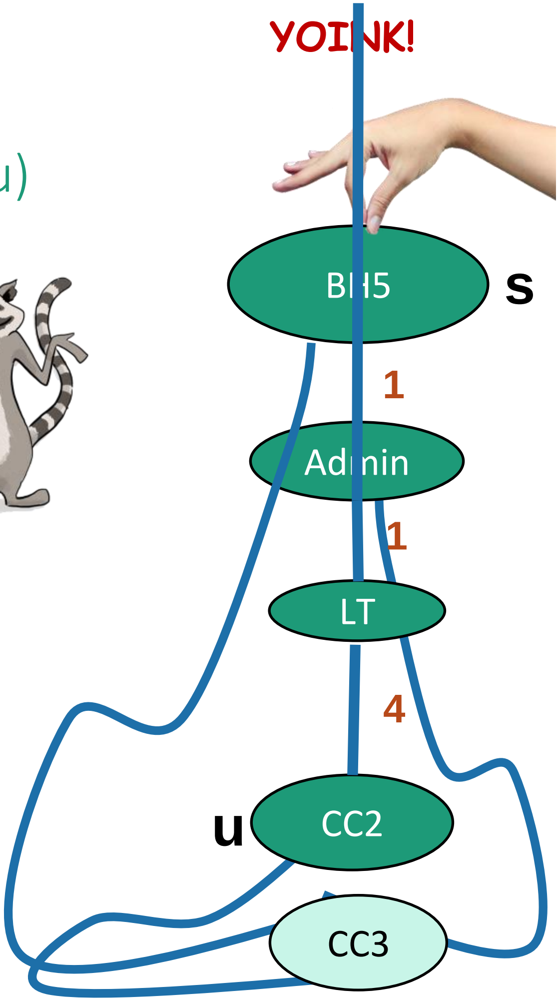
- But we should actually prove it.

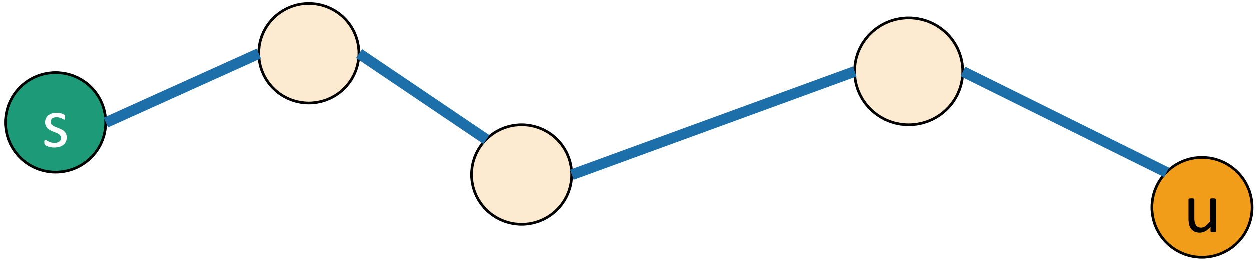


Claim 2

Inductive step

Temporary definition:
v is "good" means that d[v] = d(s,v)

- Want to show that u is good.
- Consider a **true** shortest path from s to u:



The vertices in between may or may not be **sure.**

True shortest path.

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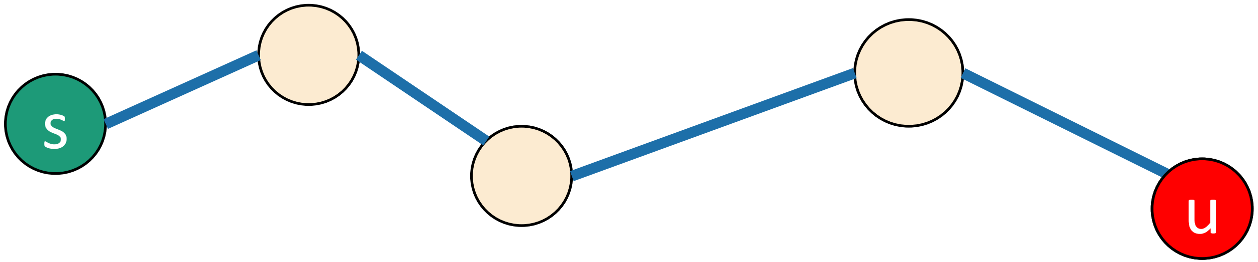
means good means not good

"by way of contradiction"

- Want to show that u is good. BWOC, suppose u isn't good.



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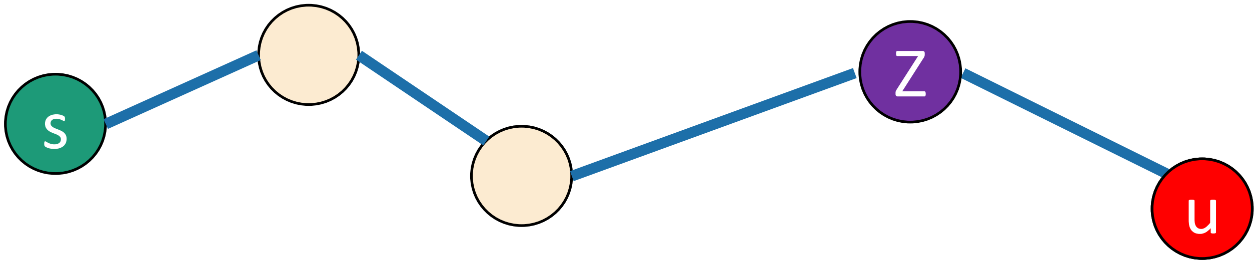
- Want to show that u is good. BWOC, suppose u isn't good.
- Say z is the good vertex before u.



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(purple circle) means good (red circle) means not good

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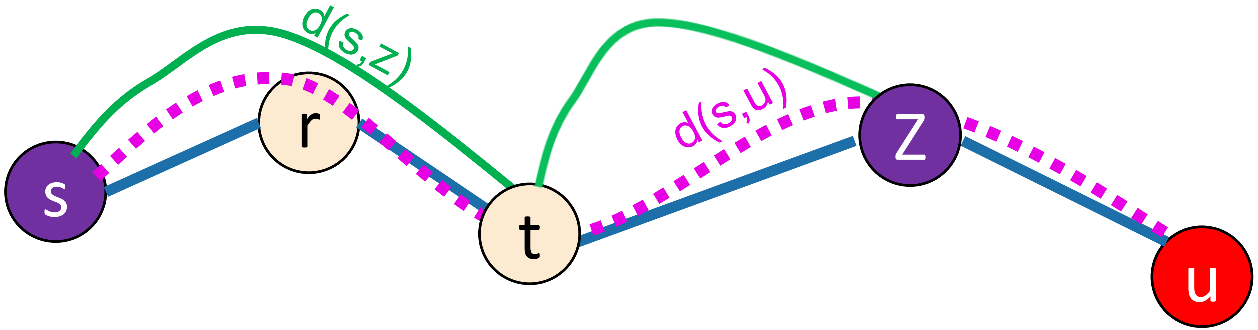
$$d[z] = d(s,z) \leq d(s,u) \leq d[u]$$

z is good | Subpaths of shortest paths are shortest paths. | Claim 1



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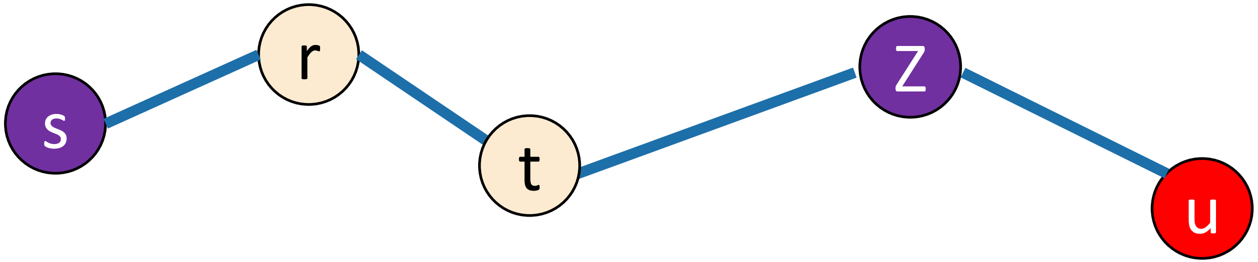

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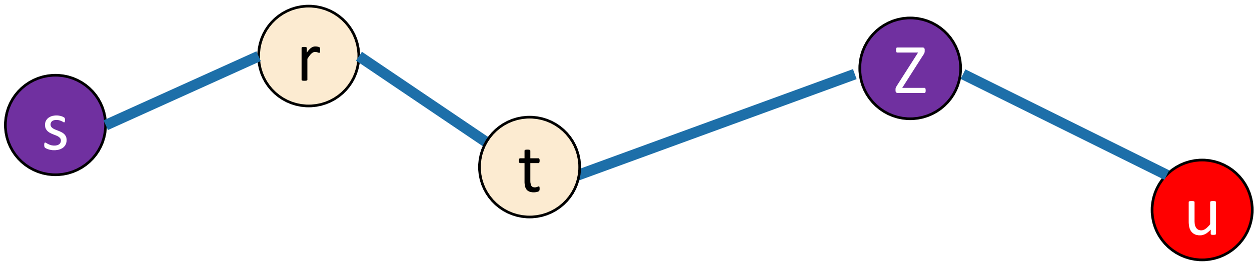

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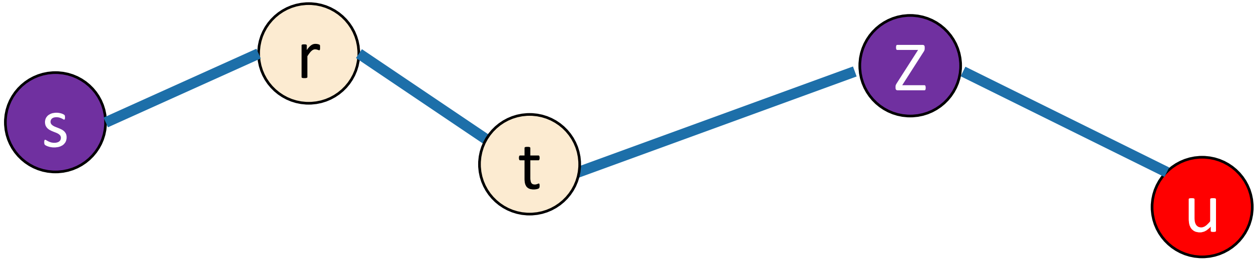
- If $d[z] = d[u]$, then u is good.
- So $d[z] < d[u]$, so z is **sure.**

But u is not good!

We chose u so that d[u] was smallest of the unsure vertices.



Claim 2

Inductive step

Temporary definition:

v is "good" means that d[v] = d(s,v)

(purple) means good (red) means not good

- Want to show that u is good. BWOC, suppose u isn't good.

$$d[z] = d(s,z) \leq d(s,u) \leq d[u]$$

z is good — Subpaths of shortest paths are shortest paths. — Claim 1

- If $d[z] = d[u]$, then u is good.
- So $d[z] < d[u]$, so z is **sure.**

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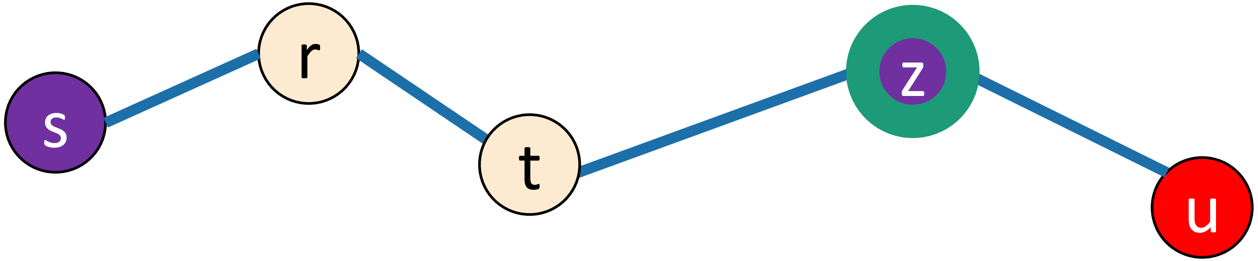

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- If z is sure then we've already updated u:

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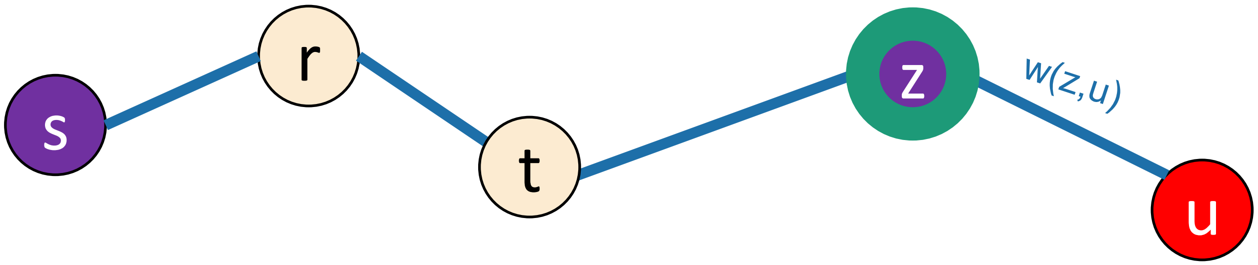

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means good

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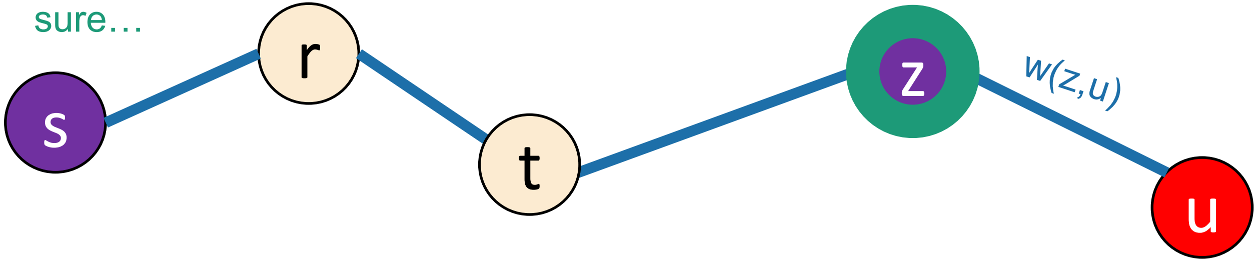
- $d[u] \leq d[z] + w(z,u)$ def of update $d[u] \leftarrow min\{ d[u], d[z] + w(z,u)\}$

That is, the value of d[z] when z was marked sure…



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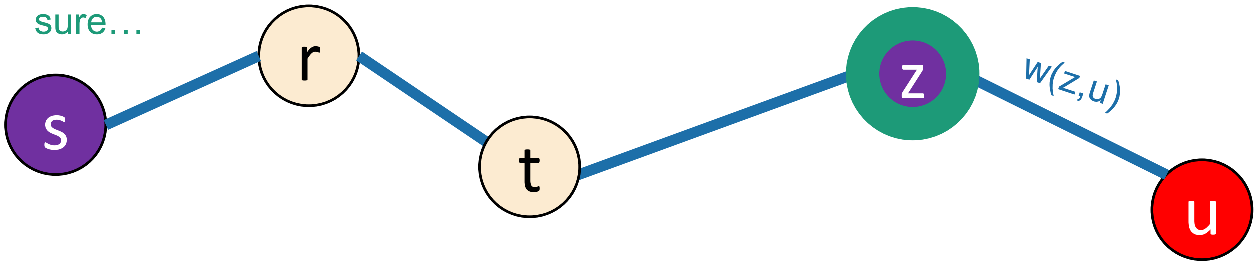

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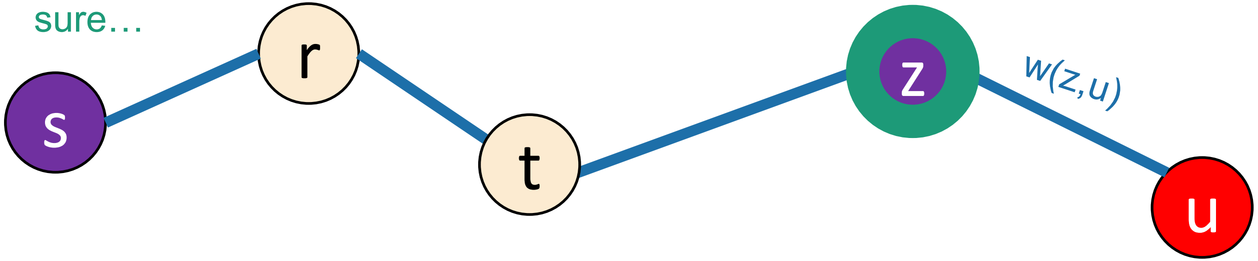

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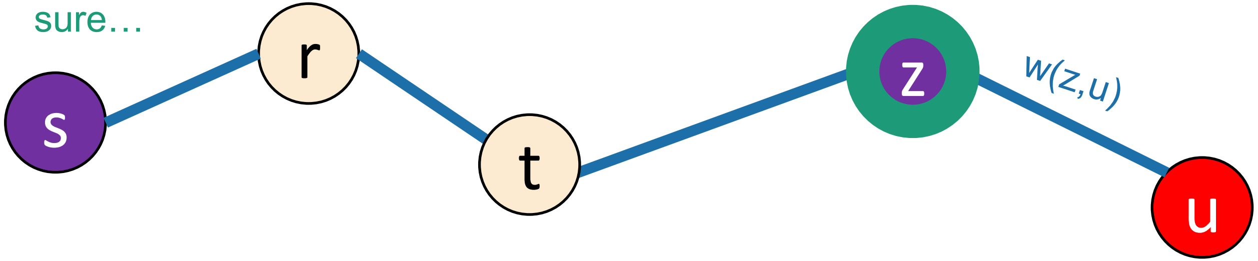

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So d(s, u) = d[u] and so u is good.

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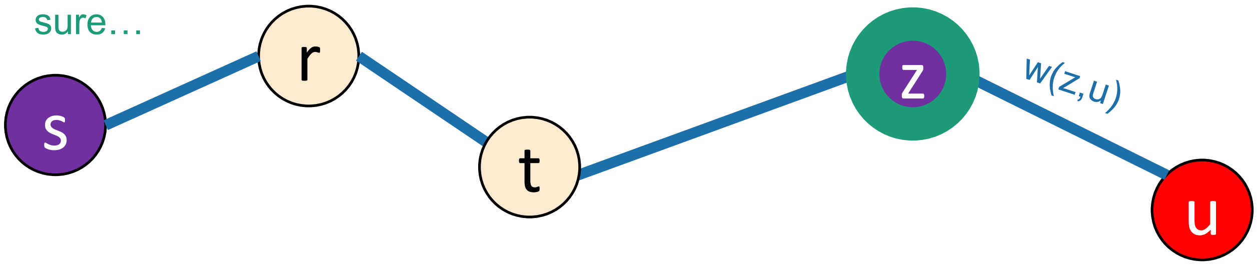

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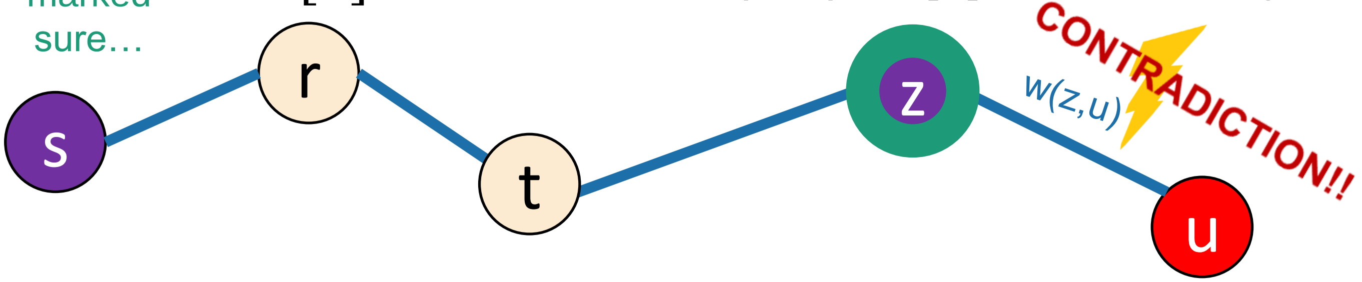

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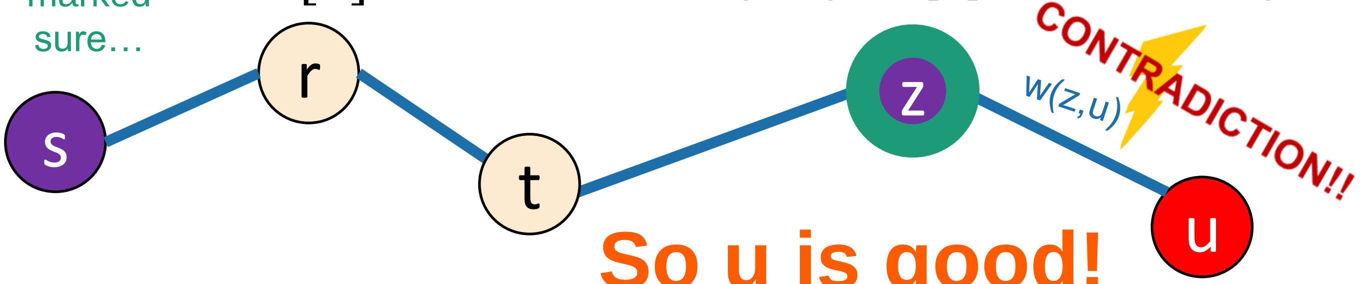

So u is good!

Back to this slide

Claim 2

When a vertex u is marked sure, d[u] = d(s,u)

- Inductive Hypothesis:
  - When we mark the t^{th} vertex v as sure, d[v] = d(s,v).
- Base case:
  - The first vertex marked **sure** is s, and d[s] = d(s,s) = 0.
- Inductive step:
  - Suppose that we are about to add u to the **sure** list.
  - That is, we picked u in the first line here:

> - Pick the **not-sure** node u with the smallest estimate **d[u].**
> - Update all u's neighbors v:
>   - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
> - Mark u as **sure**.
> - Repeat

  - Assume by induction that every v already marked **sure** has d[v] = d(s,v).
  - Want to show that d[u] = d(s,u).

Conclusion: Claim 2 holds!

Why does this work?

Now back to this slide

- **Theorem**:
  - Run Dijkstra on G =(V,E) starting from s.
  - At the end of the algorithm, the estimate **d[v]** is the actual distance d(s,v).

- Proof outline:
  - **Claim 1**: For all v, **d[v] $\geq$ d(s,v).**
  - **Claim 2**: When a vertex is marked **sure**, **d[v] = d(s,v).**

- **Claims 1 and 2** imply the **theorem.**

As usual

- Does it work?
  - Yes.

- Is it fast?
  - Depends on how you implement it.

Running time?

Dijkstra(G,s):

- Set all vertices to **not-sure**
- d[v] = ∞ for all v in V
- d[s] = 0
- **While** there are **not-sure** nodes:
  - Pick the **not-sure** node u with the smallest estimate **d[u].**
  - **For** v in u.neighbors:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
  - Mark u as **sure**.
- Now dist(s, v) = d[v]

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- n iterations (one per vertex)
- How long does one iteration take?

Depends on how we implement it…

We need a data structure that:

Just the inner loop:

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Total running time is big-oh of:

$$\sum_{u \in V} \left(T(\text{findMin}) + \left(\sum_{v \in u.neighbors} T(\text{updateKey}) \right) + T(\text{removeMin}) \right)$$

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= n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`)

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- T(findMin) = O(n)
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  = $O(n^2) + O(m)$

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  $= O(n(\,T(\texttt{findMin}) + T(\texttt{removeMin})\,) + m\,T(\texttt{updateKey}))$

  $= O(n\log(n)) + O(m\log(n))$

  $= O((n + m)\log(n))$

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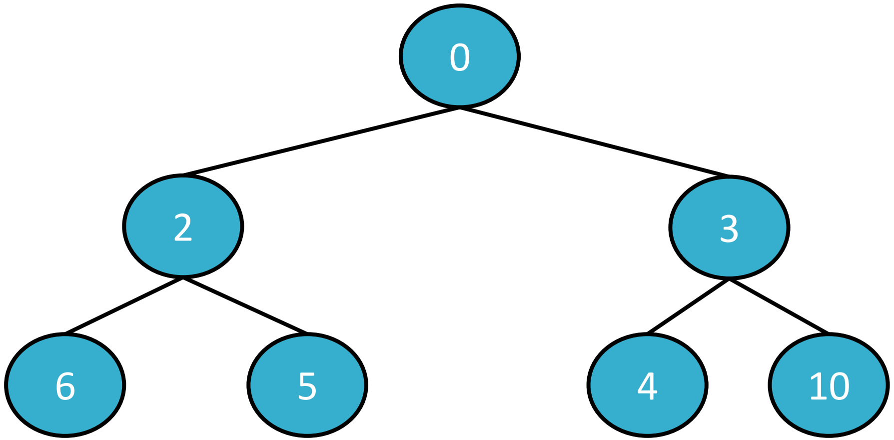
$$= O(n\log(n)) + O(m\log(n))$$
$$= O((n + m)\log(n))$$

Better than an array if the graph is sparse!

aka if m is much smaller than n^2

Heaps support these operations

- T(findMin)
- T(removeMin)
- T(updateKey)



- A **heap** is a tree-based data structure that has the property that every node has a smaller key than its children.

Many heap implementations

Nice chart on Wikipedia:

Operation	Binary[7]	Leftist	Binomial[7]	Fibonacci[7][8]	Pairing[9]	Brodal[10][b]	Rank-pairing[12]	Strict Fibonacci[13]
find-min	$\Theta(1)$	$\Theta(1)$	$\Theta(\log n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
delete-min	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$[c]	$O(\log n)$[c]	$O(\log n)$	$O(\log n)$[c]	$O(\log n)$
insert	$O(\log n)$	$\Theta(\log n)$	$\Theta(1)$[c]	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
decrease-key	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$[c]	$o(\log n)$[c][d]	$\Theta(1)$	$\Theta(1)$[c]	$\Theta(1)$
merge	$\Theta(n)$	$\Theta(\log n)$	$O(\log n)$[e]	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$

Say we use a **Fibonacci Heap**

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- Running time of Dijkstra

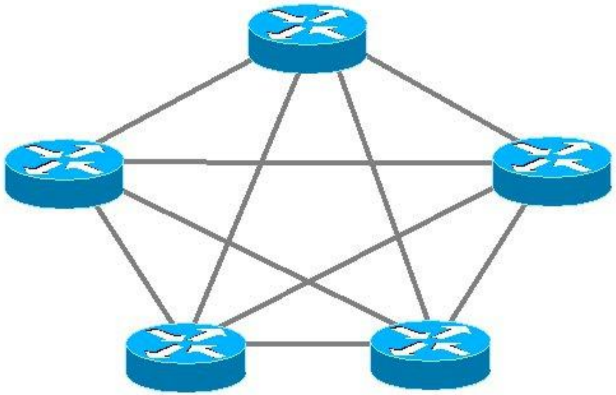
  =O(n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`))

  =O(nlog(n) + m)

Dijkstra is used in practice

- eg, OSPF (Open Shortest Path First), a routing protocol for IP networks, uses Dijkstra.

But there are some things it's not so good at.



Dijkstra Drawbacks

- Needs non-negative edge weights.
- If the weights change, we need to re-run the whole thing.
  - in OSPF, a vertex broadcasts any changes to the network, and then every vertex re-runs Dijkstra's algorithm from scratch.

Summary

- BFS:
  - (+) O(n+m)
  - (-) only unweighted graphs

- Dijkstra's algorithm:
  - (+) weighted graphs
  - (+) O(nlog(n) + m) if you implement it right.
  - (-) no negative edge weights
  - (-) very "centralized" (need to keep track of all the vertices to know which to update).

Acknowledgement

- Stanford University

Thank You