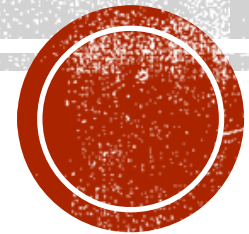**Indian Institute of Information Technology Allahabad**

Data Structures and Algorithms

Heap Sort and Priority Queue

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Priority Queue ADT

◆ A priority queue stores a collection of entries

◆ Each **entry** is a pair (key, value)

◆ Main methods of the Priority Queue ADT
  - insert(k, x) inserts an entry with key k and value x

  - removeMin() removes and returns the entry with smallest key

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- Main methods of the Priority Queue ADT
  - insert(k, x) inserts an entry with key k and value x
  - removeMin() removes and returns the entry with smallest key

- Additional methods
  - min() returns, but does not remove, an entry with smallest key
  - size(), isEmpty()

- Applications:
  - Standby flyers
  - Auctions
  - Stock market

4

Implementing Priority Queue with Linked Lists

◆ Implementation with an unsorted list

◆ Performance:

- insert takes $O(1)$ time since we can insert the item at the beginning or end of the sequence

- removeMin and min take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

Implementing Priority Queue with Linked Lists

◆ Implementation with an unsorted list

(4)—(5)—(2)—(3)—(1)

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◆ Implementation with a sorted list

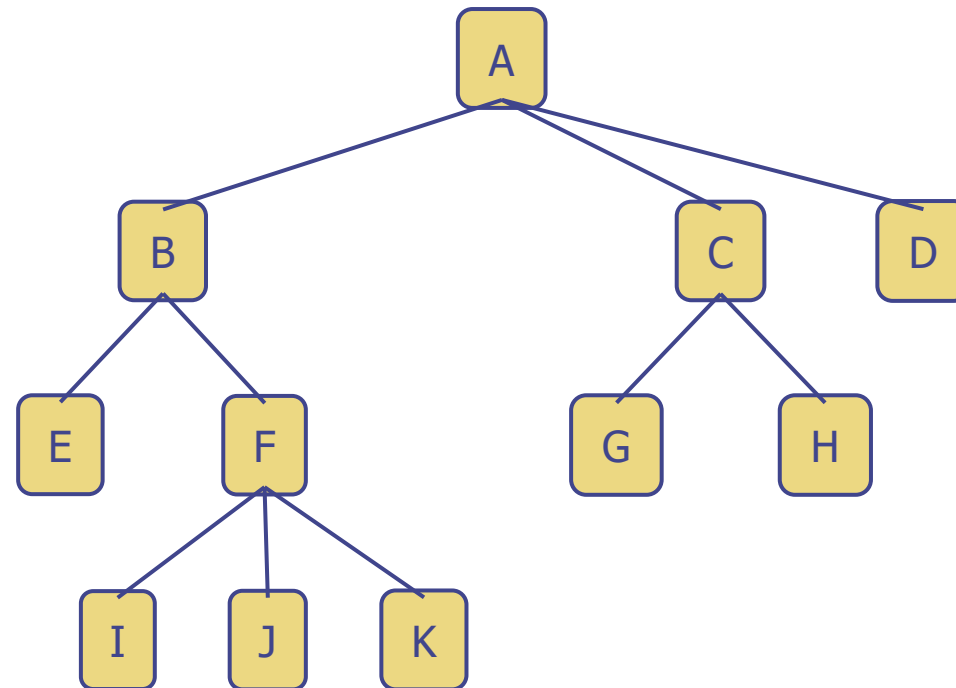(1)—(2)—(3)—(4)—(5)

◆ Performance:
- insert takes $O(n)$ time since we have to find the place where to insert the item
- removeMin and min take $O(1)$ time, since the smallest key is at the beginning

Can we do better?

Yes, using *Heaps* ,
which are built using *Trees* :

Heap Data Structure

- Array can be viewed as a nearly complete binary tree.
  - Physically – linear array.
  - Logically – binary tree, filled on all levels (except lowest.)
- Map from array elements to tree nodes and vice versa
  - Root – $A[1]$
  - Left[i] – $A[2i]$
  - Right[i] – $A[2i+1]$
  - Parent[i] – $A[\lfloor i/2 \rfloor]$

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  - Root – $A[1]$
  - Left$[i]$ – $A[2i]$
  - Right$[i]$ – $A[2i+1]$
  - Parent$[i]$ – $A[\lfloor i/2 \rfloor]$
- length$[A]$ – number of elements in array A.
- heap-size$[A]$ – number of elements in heap stored in A.
  - heap-size$[A] \leq$ length$[A]$

Heap Property (Max and Min)

- Max-Heap
  - For every node excluding the root,
    value is at most that of its parent: $A[parent[i]] \geq A[i]$
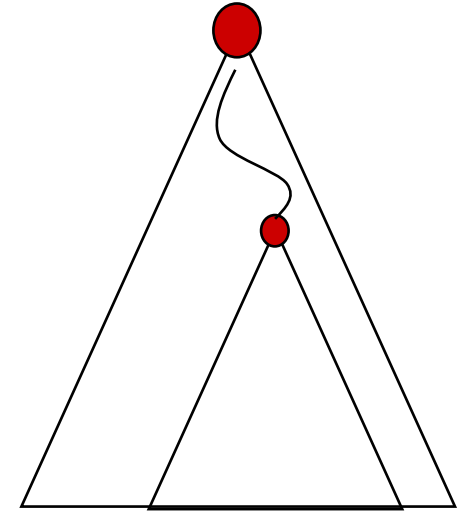- Largest element is stored at the root.
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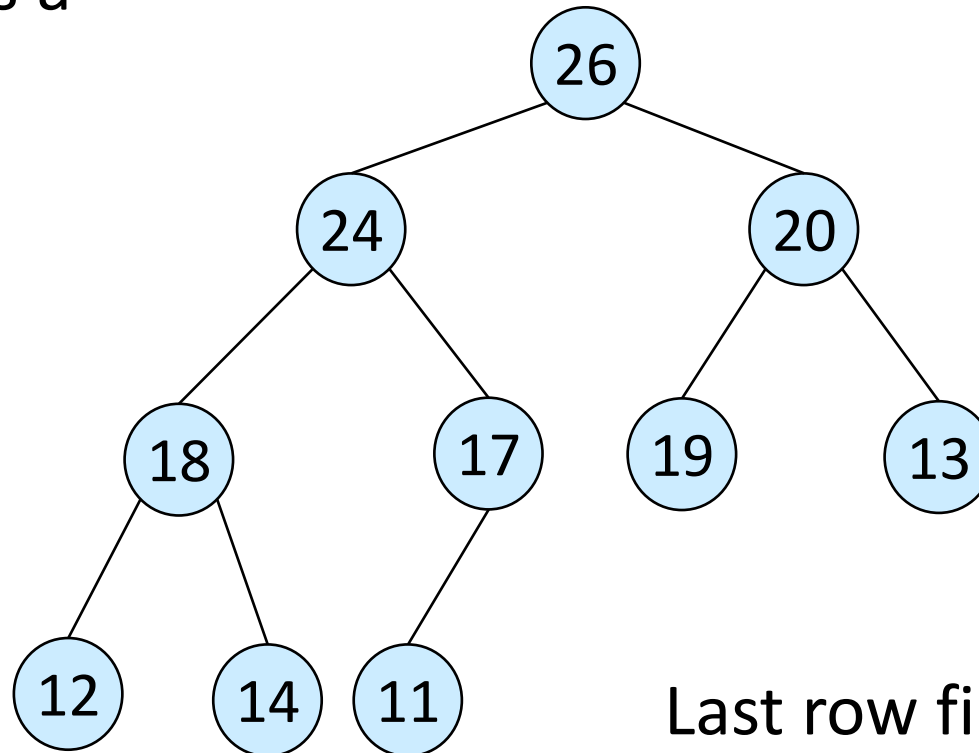- In any subtree, no values are smaller than the value stored at subtree root

Heaps – Example

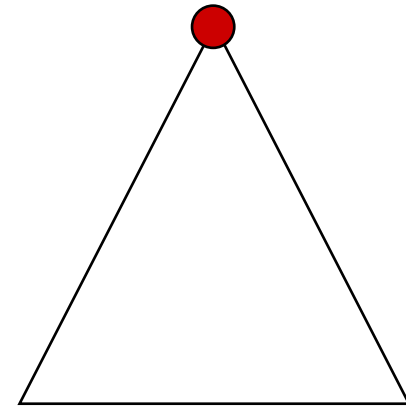| 26	24	20	18	17	19	13	12	14	11
1	2	3	4	5	6	7	8	9	10

Max-heap as an array.

Max-heap as a binary tree.

Last row filled from left to right.

Height

- *Height of a node in a tree*: the number of edges on the longest simple downward path from the node to a leaf.

- *Height of a tree*: the height of the root.

- Height of a heap: $\lfloor \log n \rfloor$

 - Basic operations on a heap run in $O(\log n)$ time

Heapsort

- Combines the better attributes of merge sort and insertion sort.
  - Like merge sort, but unlike insertion sort, running time is $O(n \lg n)$.
  - Like insertion sort, but unlike merge sort, sorts in place.
- Introduces an algorithm design technique
  - Create data structure (*heap*) to manage information during the execution of an algorithm.
- The *heap* has other applications beside sorting.
  - Priority Queues

Heaps in Sorting

- Use max-heaps for sorting.
- The array representation of max-heap is not sorted.
- Steps in sorting
  - Convert the given array of size n to a max-heap (*BuildMaxHeap*)
  - Swap the first and last elements of the array.
    - Now, the largest element is in the last position – where it belongs.
    - That leaves $n – 1$ elements to be placed in their appropriate locations.
    - However, the array of first $n – 1$ elements is no longer a max-heap.
    - Float the element at the root down one of its subtrees so that the array remains a max-heap (MaxHeapify)
    - Repeat step 2 until the array is sorted.

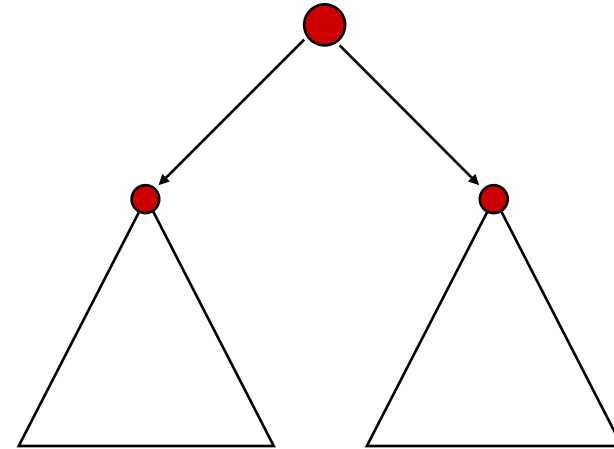# Heap Characteristics

- *Height* $\qquad = \lfloor \log n \rfloor$ i.e., floor(log n)
- No. of *leaves* $\qquad = \lceil n/2 \rceil$ i.e., ceil(log n)
- No. of nodes of height h $\leq \lceil n/2^{h+1} \rceil$
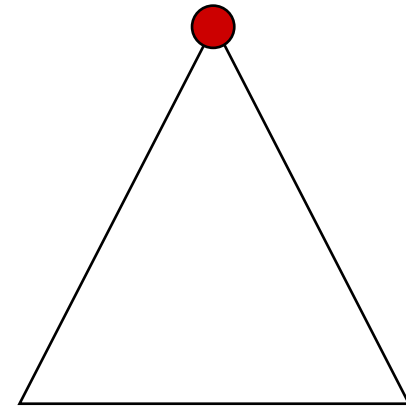
Maintaining the heap property
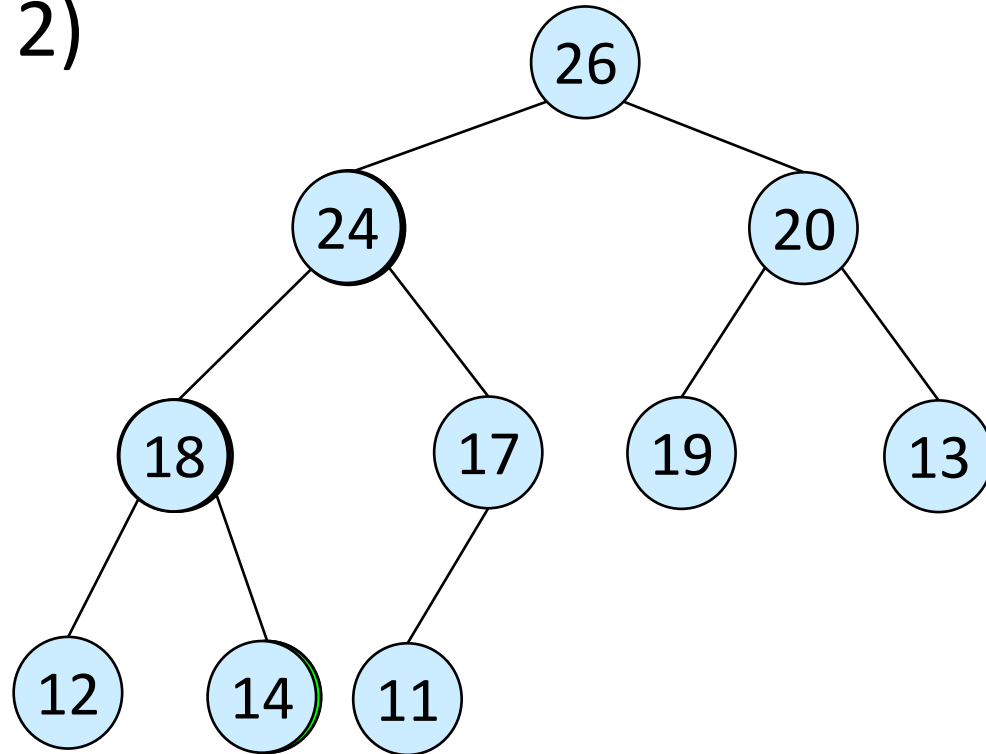
- Suppose two subtrees are max-heaps, but the root violates the max-heap property.

- Fix the offending node by exchanging the value at the node with the larger of the values at its children.

  - May lead to the subtree at the child not being a heap.

- Recursively fix the children until all of them satisfy the max-heap property.

MaxHeapify – Example

MaxHeapify(*A*, 2)

Procedure MaxHeapify

MaxHeapify(A, i)

1. $l = \text{left}(i)$

2. $r = \text{right}(i)$

3. **if** $l \leq heap\text{-}size[A]$ and $A[l] > A[i]$

4.   **then** $largest = l$

5.   **else** $largest = i$

6. **if** $r \leq heap\text{-}size[A]$ **and** $A[r] > A[largest]$

7.   **then** $largest = r$

8. **if** $largest \neq i$

9.   **then** exchange $A[i] \leftrightarrow A[largest]$

10.   *MaxHeapify(A, largest)*

Assumption:
Left(*i*) and Right(*i*)
are max-heaps.

Procedure MaxHeapify

MaxHeapify(*A*, *i*)

1. l = left(i)

2. r = right(i)

3. **if** $l \leq$ *heap-size*[A] and $A[l] > A[i]$

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8. **if** *largest* $\neq$ i

9.   **then** exchange $A[i] \leftrightarrow A[largest]$

10.   *MaxHeapify*(*A*, *largest*)

Time to fix node *i* and its children = $\Theta(1)$

PLUS

Time to fix the subtree rooted at one of *i*'s children = T(size of subtree at *largest*)

Running Time for MaxHeapify(A, n)

- MaxHeapify takes $O(h)$ where h is the height of the node where MaxHeapify is applied

- Alternately, $T(n) = O(\log n)$ in worst case

Building a heap

- Use *MaxHeapify* to convert an array *A* into a max-heap.
- How?

Building a heap

- Use *MaxHeapify* to convert an array *A* into a max-heap.
- How?
- Call MaxHeapify on each element in a bottom-up manner.

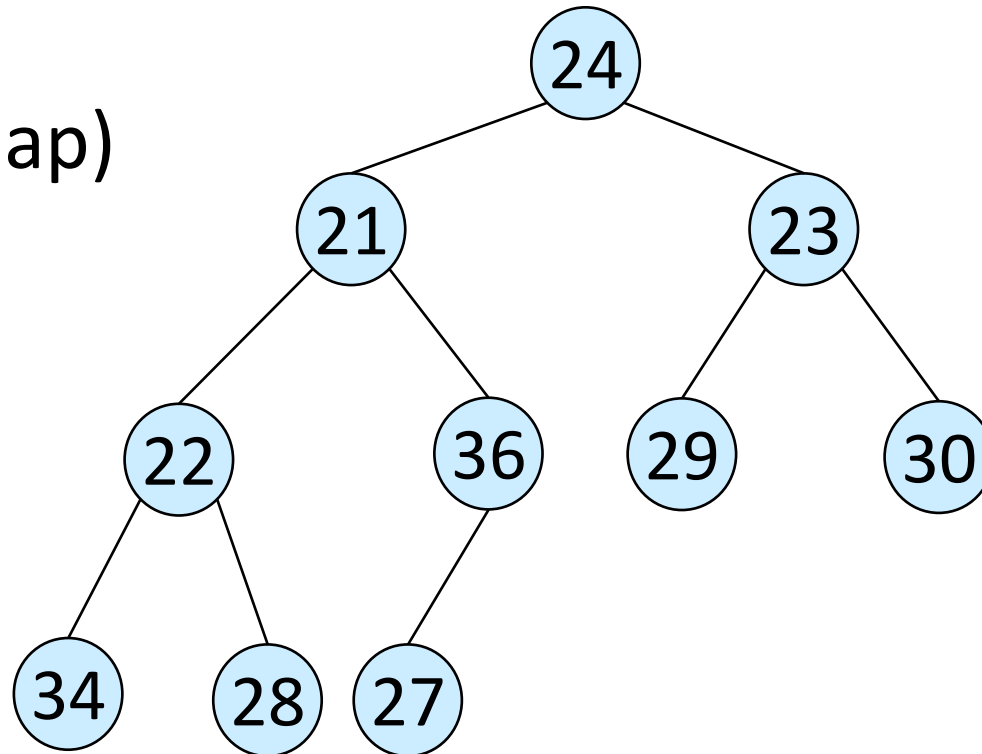$\underline{BuildMaxHeap(A)}$

1. $heap\text{-}size[A] = length[A]$
2. **for** $i = \lfloor length[A]/2 \rfloor$ **downto** 1
3.   **do** $MaxHeapify(A, i)$

BuildMaxHeap – Example

Input Array:

24	21	23	22	36	29	30	34	28	27

Initial Heap:
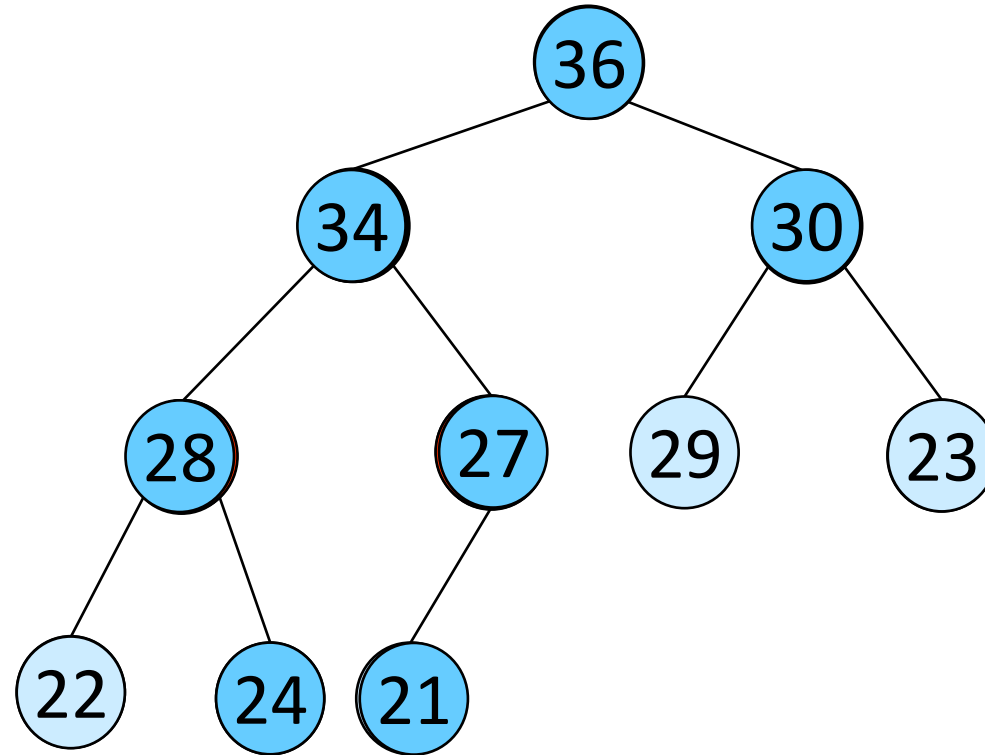(<u>not</u> max-heap)

BuildMaxHeap – Example

MaxHeapify($\lfloor 10/2 \rfloor = 5$)
MaxHeapify(4)
MaxHeapify(3)
MaxHeapify(2)
MaxHeapify(1)

Correctness of *BuildMaxHeap*

- Loop Invariant: At the start of each iteration of the **for** loop, each node $i+1, i+2, \ldots, n$ is the root of a max-heap.

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  - Before first iteration $i = \lfloor n/2 \rfloor$
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Correctness of *BuildMaxHeap*

- <u>Loop Invariant:</u> At the start of each iteration of the **for** loop, each node $i+1, i+2, ..., n$ is the root of a max-heap.

- <u>Initialization:</u>
  - Before first iteration $i = \lfloor n/2 \rfloor$
  - Nodes $\lfloor n/2 \rfloor+1, \lfloor n/2 \rfloor+2, ..., n$ are leaves and hence roots of max-heaps.

- <u>Maintenance:</u>
  - By Loop Invariant, subtrees at children of node i are max heaps.
  - Hence, MaxHeapify(i) renders node i a max heap root (while preserving the max heap root property of higher-numbered nodes).
  - Decrementing i reestablishes the loop invariant for the next iteration.

Running Time of *BuildMaxHeap*

- Loose upper bound:

  - Cost of a *MaxHeapify* call $\times$ No. of calls to *MaxHeapify*

  - $O(\log n) \times O(n) = O(n \log n)$

- Tighter bound:

  - Cost of a call to *MaxHeapify* at a node depends on the height, h, of the node – $O(h)$.

  - Height of most nodes smaller than n.

  - Height of nodes h ranges from 0 to $\lfloor \log n \rfloor$.

  - No. of nodes of height h is $\lceil n/2^{h+1} \rceil$

Running Time of *BuildMaxHeap*

Tighter Bound for *T*(*BuildMaxHeap*)

T(*BuildMaxHeap*)

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

$$= O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

Running Time of *BuildMaxHeap*

Tighter Bound for *T(BuildMaxHeap)*

T(BuildMaxHeap)

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

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$$\sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}$$

$$\leq \sum_{h=0}^{\infty} \frac{h}{2^h}$$

$$= \frac{1/2}{(1-1/2)^2}$$

$$= 2$$

Running Time of *BuildMaxHeap*

Tighter Bound for $T(BuildMaxHeap)$

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$$= O(n)$$

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$$\sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}$$

$$\leq \sum_{h=0}^{\infty} \frac{h}{2^h}$$

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$$= 2$$

Can build a heap from an unordered array in linear time

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Heapsort

- Sort by maintaining as yet unsorted elements as a max-heap.
- Start by building a max-heap on all elements in A.
  - Maximum element is in the root, $A[1]$.
- Move the maximum element to its correct final position.
  - Exchange $A[1]$ with $A[n]$.
- Discard $A[n]$ – it is now sorted.
  - Decrement heap-size$[A]$.
- Restore the max-heap property on $A[1..n-1]$.
  - Call *MaxHeapify*$(A, 1)$.
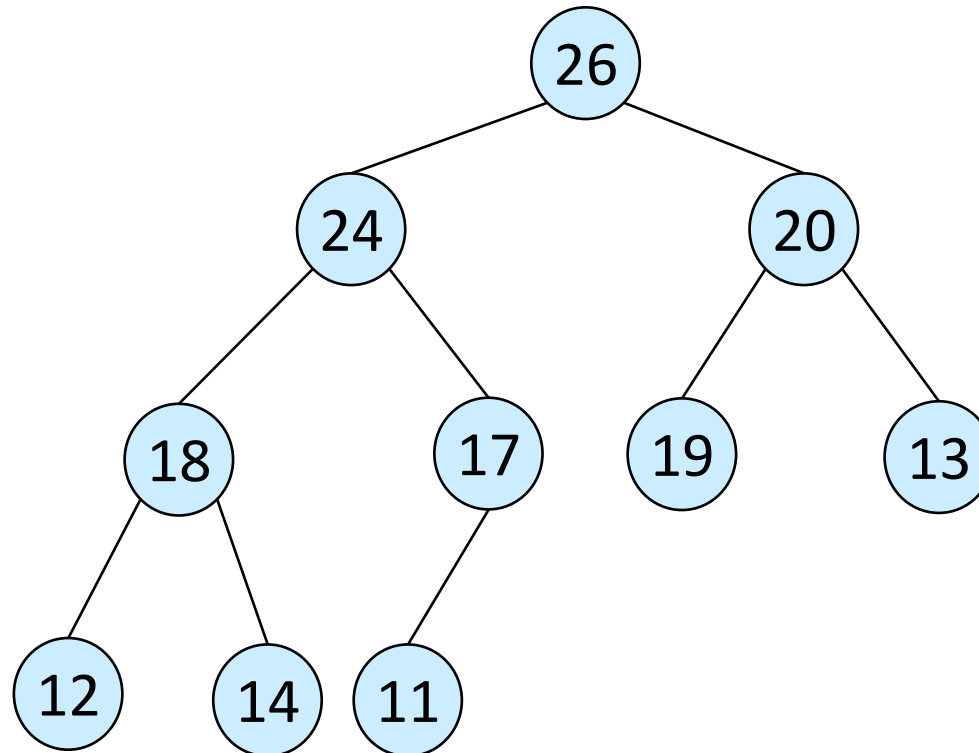- Repeat until heap-size$[A]$ is reduced to 2.

Heapsort(A)

HeapSort(*A*)

1. Build-Max-Heap(*A*)
2. **for** *i* = *length*[*A*] **downto** 2
3.   **do** exchange *A*[1] ↔ *A*[*i*]
4.   *heap-size*[*A*] = *heap-size*[*A*] − 1
5.   *MaxHeapify*(*A*, 1)

Heapsort – Example

26	24	20	18	17	19	13	12	14	11
1	2	3	4	5	6	7	8	9	10

Algorithm Analysis

- In-place

- Not Stable

- Build-Max-Heap takes $O(n)$ and each of the $n-1$ calls to Max-Heapify takes time $O(\log n)$.

- Therefore, $T(n) = O(n \log n)$

$\underline{HeapSort(A)}$

1. Build-Max-Heap(A)
2. **for** $i = length[A]$ **downto** 2
3.   **do** exchange $A[1] \leftrightarrow A[i]$
4. $heap\text{-}size[A] = heap\text{-}size[A] - 1$
5. $MaxHeapify(A, 1)$

Heap Procedures for Sorting

- MaxHeapify $\quad\quad O(\log n)$

- BuildMaxHeap $\quad O(n)$

- HeapSort $\quad\quad\quad O(n \log n)$

Priority Queue

- Popular & important application of heaps.
- Max and min priority queues.
- Maintains a *dynamic* set *S* of elements.
- Each set element has a *key* – an associated value.
- Goal is to support insertion and extraction efficiently.

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- Each set element has a *key* – an associated value.
- Goal is to support insertion and extraction efficiently.
- Applications:
  - Ready list of processes in operating systems by their priorities – the list is highly dynamic
  - In event-driven simulators to maintain the list of events to be simulated in order of their time of occurrence.

Basic Operations

- Operations on a max-priority queue:
  - Insert(*S, x*) - inserts the element *x* into the set *S*
    - $S \leftarrow S \cup \{x\}$.
  - Maximum(*S*) - returns the element of *S* with the largest key.
  - Extract-Max(*S*) - removes and returns the element of *S* with the largest key.
  - Increase-Key(*S, x, k*) – increases the value of element *x*'s key to the new value *k*.

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- Operations on a max-priority queue:
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    - Extract-Max(*S*) - removes and returns the element of *S* with the largest key.
    - Increase-Key(*S, x, k*) – increases the value of element *x*'s key to the new value *k*.
- Min-priority queue supports Insert, Minimum, Extract-Min, and Decrease-Key.
- Heap gives a good compromise between fast insertion but slow extraction and vice versa.

Heap Property (Max and Min)

- Max-Heap
  - For every node excluding the root,
    value is at most that of its parent: $A[parent[i]] \geq A[i]$
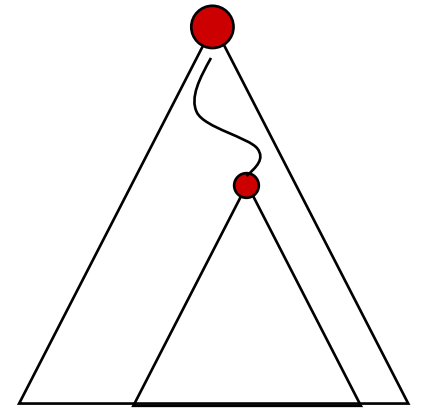- Largest element is stored at the root.
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- Min-Heap
  - For every node excluding the root,
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- Smallest element is stored at the root.
- In any subtree, no values are smaller than the value stored at subtree root

Heap-Extract-Max(A)

Implements the Extract-Max operation.

Heap-Extract-Max(A)

1. if *heap-size*[*A*] < 1
2.   then error "heap underflow"
3. *max* = *A*[1]
4. *A*[1] = *A*[*heap-size*[*A*]]
5. *heap-size*[*A*] = *heap-size*[*A*] - 1
6. MaxHeapify(*A*, 1)
7. return max

Running time :
Dominated by the running time of MaxHeapify
= *O*(log *n*)

Heap-Insert(A, key)

Heap-Insert(A, key)

1. *heap-size*[A] = *heap-size*[A] + 1
2. *i* = *heap-size*[A]
4. **while** *i* > 1 **and** *A*[Parent(*i*)] < *key*
5.   **do** *A*[*i*] = *A*[Parent(*i*)]
6.   *i* = Parent(*i*)
7. *A*[*i*] = *key*

Running time is $O(\log n)$

The path traced from the new leaf to the root has length $O(\log n)$

Heap-Increase-Key(A, i, key)

Heap-Increase-Key(A, i, key)
1   **If** $key < A[i]$
2   **then error** "new key is smaller than the current key"
3   $A[i] = key$
4   **while** $i > 1$ **and** $A[\text{Parent}[i]] < A[i]$
5   **do** exchange $A[i] \leftrightarrow A[\text{Parent}[i]]$
6              $i = \text{Parent}[i]$

Heap-Insert(A, key)
1   $heap\text{-}size[A] = heap\text{-}size[A] + 1$
2   $A[heap\text{-}size[A]] = -\infty$
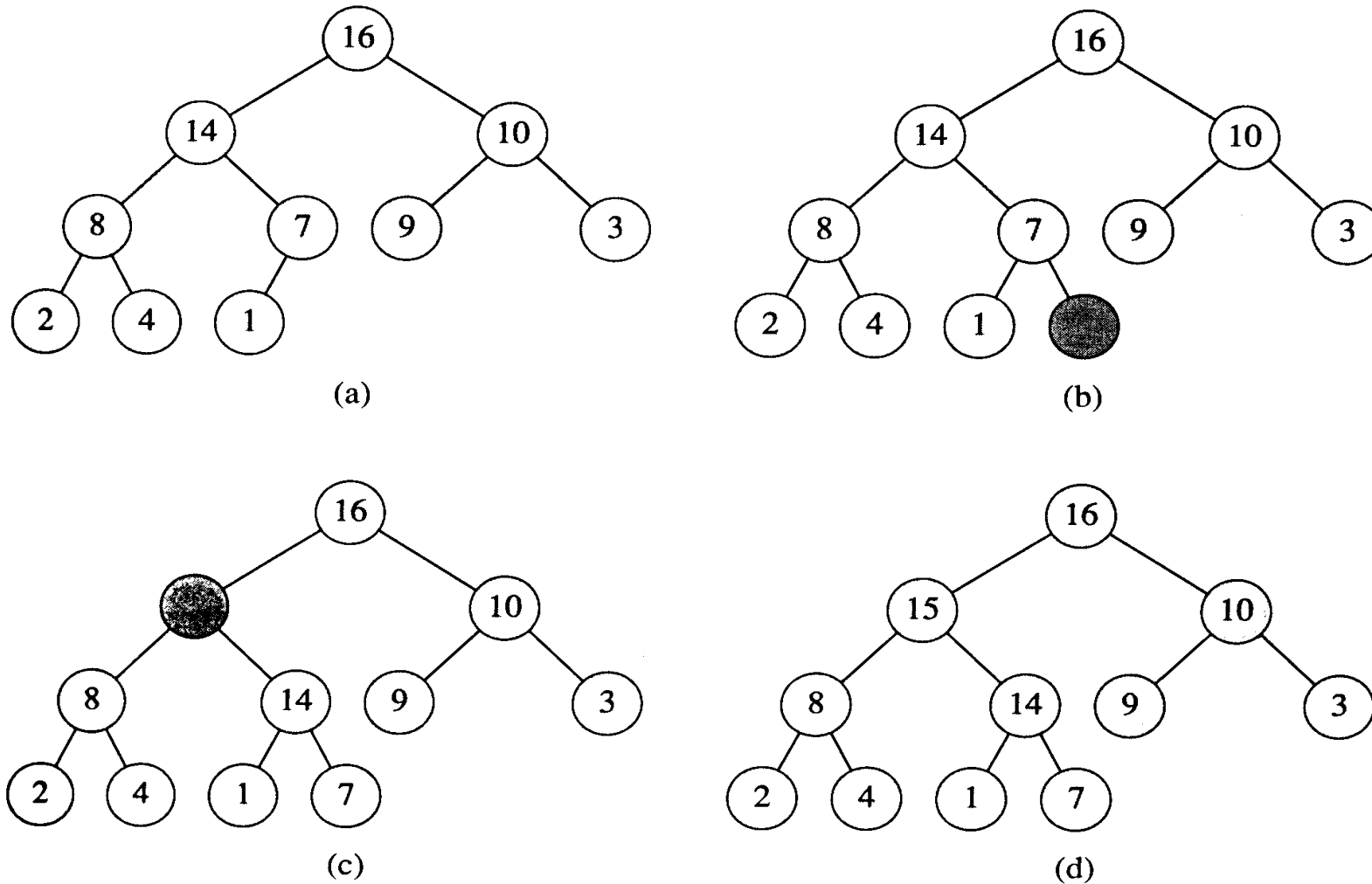3   *Heap-Increase-Key*(A, heap-size[A], key)

Examples

Figure 7.5 The operation of HEAP-INSERT. (a) The heap of Figure 7.4(a) before we insert a node with key 15. (b) A new leaf is added to the tree. (c) Values on the path from the new leaf to the root are copied down until a place for the key 15 is found. (d) The key 15 is inserted.

Acknowledgement

- University of North Carolina at Chapel Hill

Thank You