Indian Institute of Information Technology Allahabad

Data Structures

Heap Sort and Priority Queue

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Priority Queue ADT

- A priority queue stores a collection of entries
- Each **entry** is a pair (key, value)
- Main methods of the Priority Queue ADT
  - insert(k, x) inserts an entry with key k and value x
  - removeMin() removes and returns the entry with smallest key

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- Main methods of the Priority Queue ADT
  - insert(k, x)
    inserts an entry with key k and value x
  - removeMin()
    removes and returns the entry with smallest key
- Additional methods
  - min()
    returns, but does not remove, an entry with smallest key
  - size(), isEmpty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Implementing Priority Queue with Linked Lists

- Implementation with an unsorted list

- Performance:
  - insert takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
  - removeMin and min take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

Implementing Priority Queue with Linked Lists

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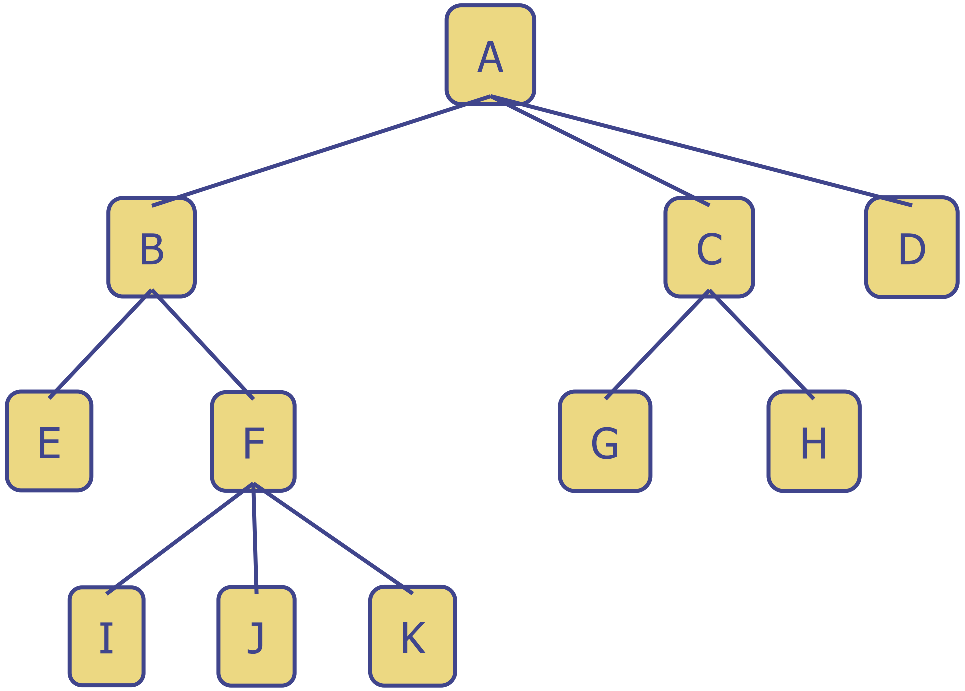
- Implementation with a sorted list

- Performance:
  - insert takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin and min take $O(1)$ time, since the smallest key is at the beginning

Can we do better?

Yes, using *Heaps* ,
which are built using *Trees* :



Heap Data Structure

- Array can be viewed as a nearly complete binary tree.
  - Physically – linear array.
  - Logically – binary tree, filled on all levels (except lowest.)
- Map from array elements to tree nodes and vice versa
  - Root – $A[1]$
  - Left[i] – $A[2i]$
  - Right[i] – $A[2i+1]$
  - Parent[i] – $A[\lfloor i/2 \rfloor]$

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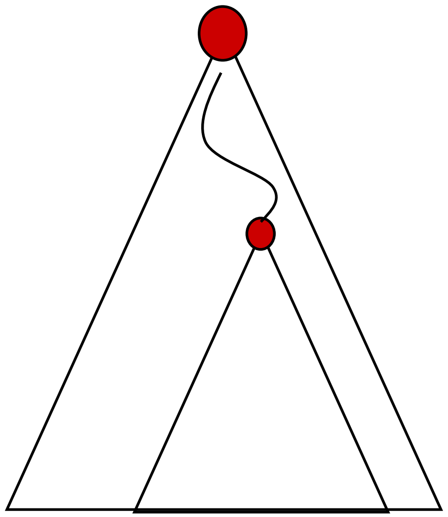
  - Root – $A[1]$
  - Left[i] – $A[2i]$
  - Right[i] – $A[2i+1]$
  - Parent[i] – $A[\lfloor i/2 \rfloor]$
- length[A] – number of elements in array A.
- heap-size[A] – number of elements in heap stored in A.
  - heap-size[A] $\leq$ length[A]

Heap Property (Max and Min)

- Max-Heap
  - For every node excluding the root, value is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
- In any subtree, no values are larger than the value stored at subtree root.



- Min-Heap
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Heaps – Example

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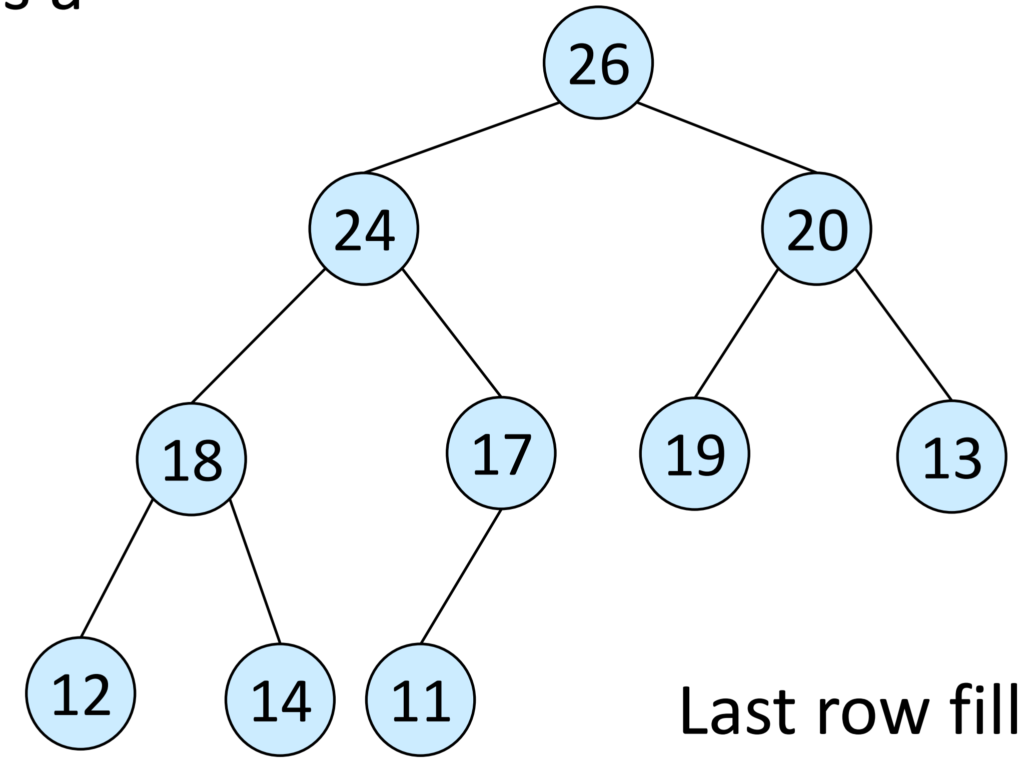
Max-heap as an array.

Max-heap as a binary tree.



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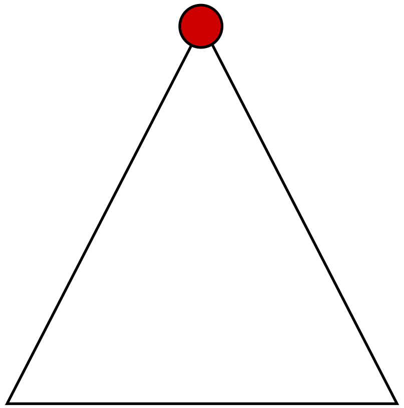
Height

- *Height of a node in a tree*: the number of edges on the longest simple downward path from the node to a leaf.
- *Height of a tree*: the height of the root.
- Height of a heap: $\lfloor \log n \rfloor$
  - Basic operations on a heap run in $O(\log n)$ time



Heapsort

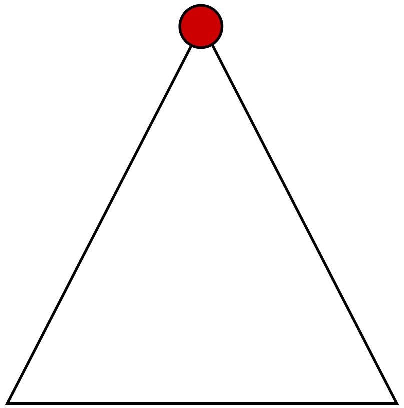
- Combines the better attributes of merge sort and insertion sort.
  - Like merge sort, but unlike insertion sort, running time is $O(n \lg n)$.
  - Like insertion sort, but unlike merge sort, sorts in place.
- Introduces an algorithm design technique
  - Create data structure (*heap*) to manage information during the execution of an algorithm.
- The *heap* has other applications beside sorting.
  - Priority Queues

Heaps in Sorting

- Use max-heaps for sorting.
- The array representation of max-heap is not sorted.
- Steps in sorting
  - Convert the given array of size *n* to a max-heap (*BuildMaxHeap*)
  - Swap the first and last elements of the array.
    - Now, the largest element is in the last position – where it belongs.
    - That leaves *n* – 1 elements to be placed in their appropriate locations.
    - However, the array of first *n* – 1 elements is no longer a max-heap.
    - Float the element at the root down one of its subtrees so that the array remains a max-heap (MaxHeapify)
    - Repeat step 2 until the array is sorted.

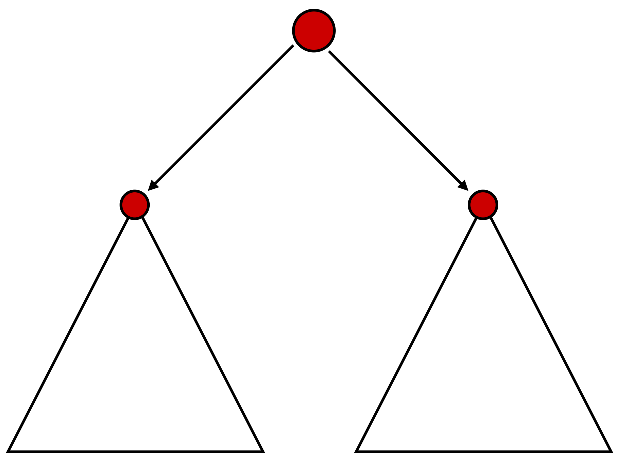
Heap Characteristics

- *Height* $= \lfloor \log n \rfloor$ i.e., floor(log n)
- No. of *leaves* $= \lceil n/2 \rceil$ i.e., ceil(log n)
- No. of nodes of height *h* $\leq \lceil n/2^{h+1} \rceil$



Maintaining the heap property

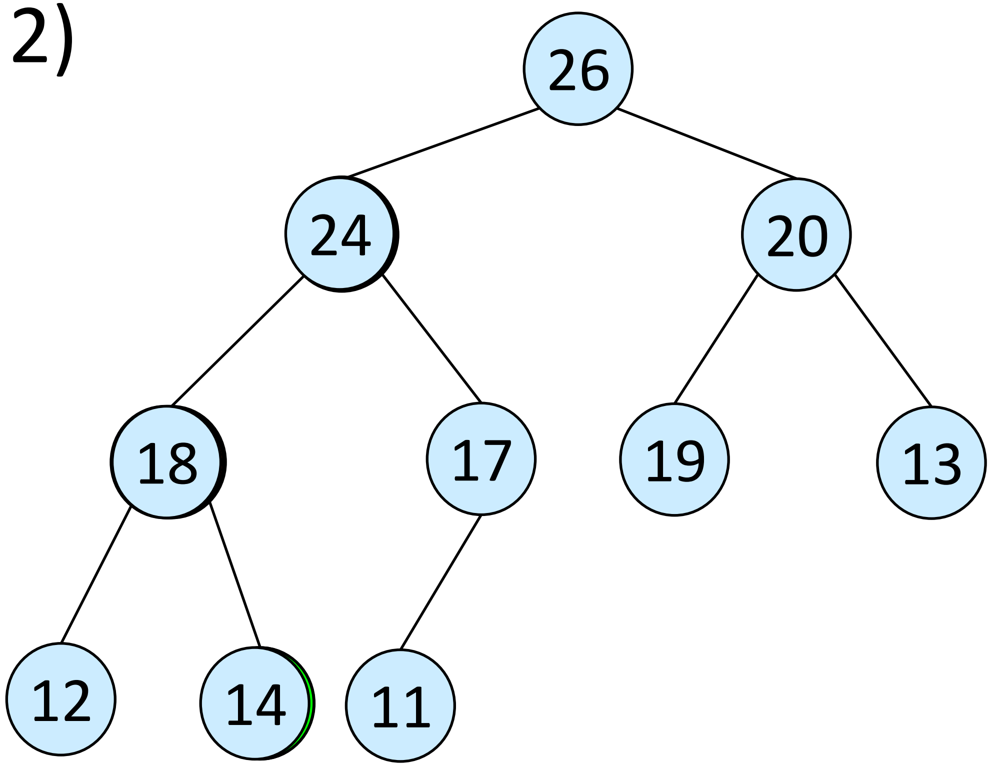
- Suppose two subtrees are max-heaps, but the root violates the max-heap property.



- Fix the offending node by exchanging the value at the node with the larger of the values at its children.
  - May lead to the subtree at the child not being a heap.
- Recursively fix the children until all of them satisfy the max-heap property.

MaxHeapify – Example

MaxHeapify(A, 2)



Procedure MaxHeapify

```
MaxHeapify(A, i)
l = left(i)
r = right(i)
if l ≤ heap-size[A] and A[l] > A[i]
    then largest = l
    else largest = i
if r ≤ heap-size[A] and A[r] > A[largest]
    then largest = r
if largest ≠ i
    then exchange A[i] ↔ A[largest]
            MaxHeapify(A, largest)
```

<u>Assumption:</u>

Left(*i*) and Right(*i*) are max-heaps.

Procedure MaxHeapify

<u>*MaxHeapify*(*A*, *i*)</u>

l = left(i)
r = right(i)
if $l \leq$ *heap-size*[A] and $A[l] > A[i]$
    then *largest* = l
    else *largest* = i
if $r \leq$ *heap-size*[A] **and** $A[r] > A[$*largest*$]$
    then *largest* = r
if *largest* $\neq i$
    then exchange $A[i] \leftrightarrow A[$*largest*$]$
        MaxHeapify(A, *largest*)

Time to fix node *i* and its children = **Θ(1)**

PLUS

Time to fix the subtree rooted at one of *i*'s children = *T*(size of subree at *largest*)

Running Time for MaxHeapify(A, n)

- MaxHeapify takes $O(h)$ where h is the height of the node where MaxHeapify is applied
- Alternately, $T(n) = O(\log n)$ in worst case

Building a heap

- Use *MaxHeapify* to convert an array *A* into a max-heap.
- How?

Building a heap

- Use *MaxHeapify* to convert an array *A* into a max-heap.
- How?
- Call MaxHeapify on each element in a bottom-up manner.

BuildMaxHeap(A)

1. *heap-size*[*A*] = *length*[*A*]
2. **for** $i = \lfloor length[A]/2 \rfloor$ **downto** 1
3. **do** *MaxHeapify*(*A*, *i*)

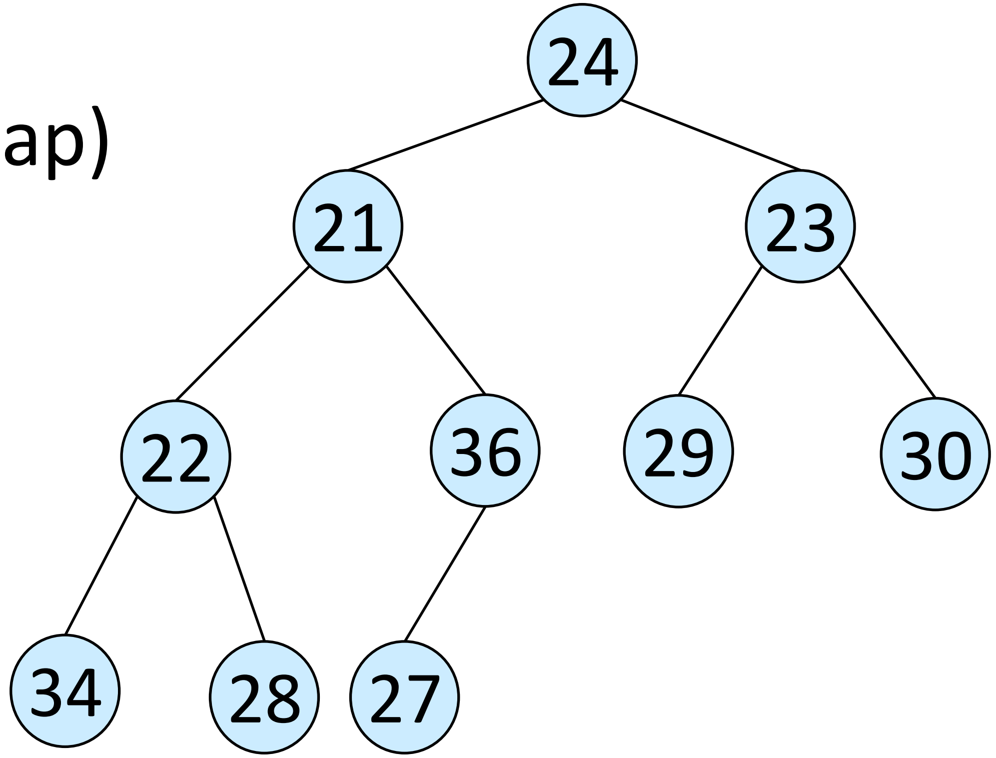
BuildMaxHeap – Example

Input Array:

24	21	23	22	36	29	30	34	28	27

Initial Heap:

(not max-heap)



BuildMaxHeap – Example

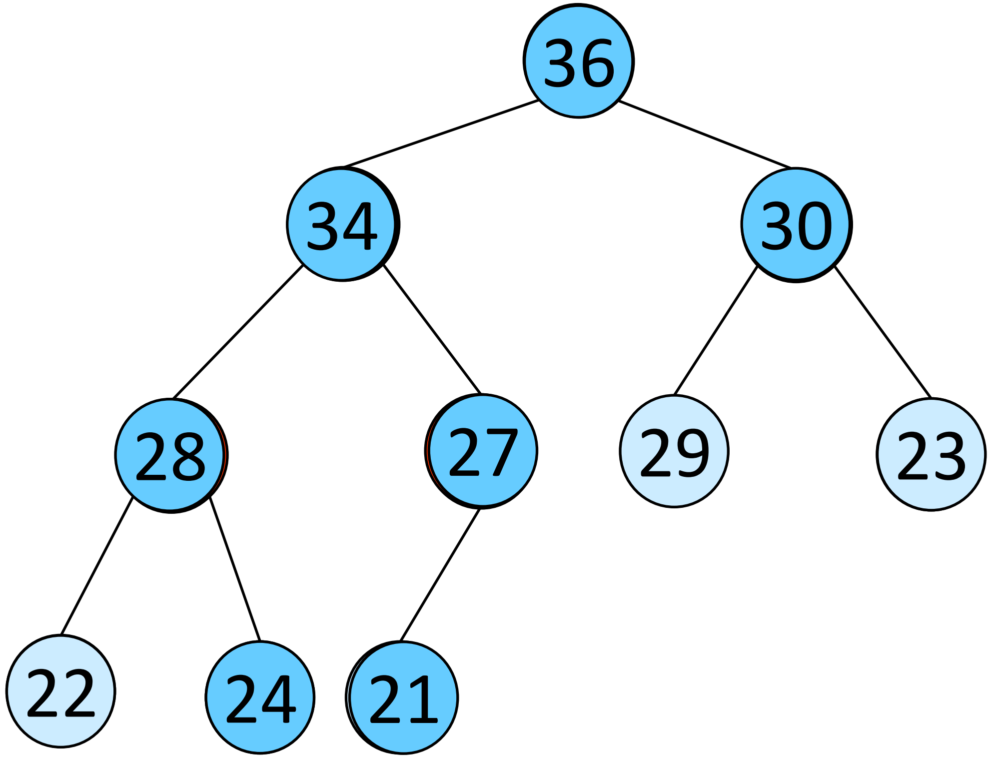
MaxHeapify($\lfloor 10/2 \rfloor = 5$)

MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

MaxHeapify(1)



Correctness of *BuildMaxHeap*

- Loop Invariant: At the start of each iteration of the **for** loop, each node i+1, i+2, …, n is the root of a max-heap.

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- <u>Loop Invariant:</u> At the start of each iteration of the **for** loop, each node i+1, i+2, …, n is the root of a max-heap.
- <u>Initialization:</u>
  - Before first iteration $i = \lfloor n/2 \rfloor$
  - Nodes $\lfloor n/2 \rfloor$+1, $\lfloor n/2 \rfloor$+2, …, n are leaves and hence roots of max-heaps.

Correctness of *BuildMaxHeap*

- Loop Invariant: At the start of each iteration of the **for** loop, each node $i+1$, $i+2$, …, n is the root of a max-heap.
- Initialization:
  - Before first iteration $i = \lfloor n/2 \rfloor$
  - Nodes $\lfloor n/2 \rfloor+1$, $\lfloor n/2 \rfloor+2$, …, n are leaves and hence roots of max-heaps.
- Maintenance:
  - By Loop Invariant, subtrees at children of node i are max heaps.
  - Hence, MaxHeapify(i) renders node i a max heap root (while preserving the max heap root property of higher-numbered nodes).
  - Decrementing i reestablishes the loop invariant for the next iteration.

Running Time of *BuildMaxHeap*

- Loose upper bound:
  - Cost of a *MaxHeapify* call $\times$ No. of calls to *MaxHeapify*
  - $O(\log n) \times O(n) = O(n\log n)$
- Tighter bound:
  - Cost of a call to *MaxHeapify* at a node depends on the height, h, of the node – $O(h)$.
  - Height of most nodes smaller than n.
  - Height of nodes h ranges from 0 to $\lfloor \log n \rfloor$.
  - No. of nodes of height h is $\lceil n/2^{h+1} \rceil$

Running Time of *BuildMaxHeap*

Tighter Bound for *T*(*BuildMaxHeap*)

T(*BuildMaxHeap*)

$$\boxed{\begin{aligned}&\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) \\ &= O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)\end{aligned}}$$

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$$\sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \le \sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2$$

Running Time of *BuildMaxHeap*

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$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

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$$\le \sum_{h=0}^{\infty} \frac{h}{2^h}$$

$$= \frac{1/2}{(1-1/2)^2}$$

$$= 2$$

$$O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$

$$= O(n)$$

Running Time of *BuildMaxHeap*

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$$\sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}$$

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$$O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$

$$= O(n)$$

Can build a heap from an unordered array in linear time

Heapsort

- Sort by maintaining as yet unsorted elements as a max-heap.
- Start by building a max-heap on all elements in *A.*
    - Maximum element is in the root, $A[1]$.
- Move the maximum element to its correct final position.
    - Exchange $A[1]$ with $A[n]$.
- Discard $A[n]$ – it is now sorted.
    - Decrement heap-size$[A]$.
- Restore the max-heap property on $A[1..n{-}1]$.
    - Call *MaxHeapify*$(A, 1)$.
- Repeat until heap-size$[A]$ is reduced to 2.

Heapsort(A)

HeapSort(A)

1. Build-Max-Heap(A)
2. **for** i = $length[A]$ **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. $heap\text{-}size[A] = heap\text{-}size[A] - 1$
5. $MaxHeapify(A, 1)$

Heapsort – Example

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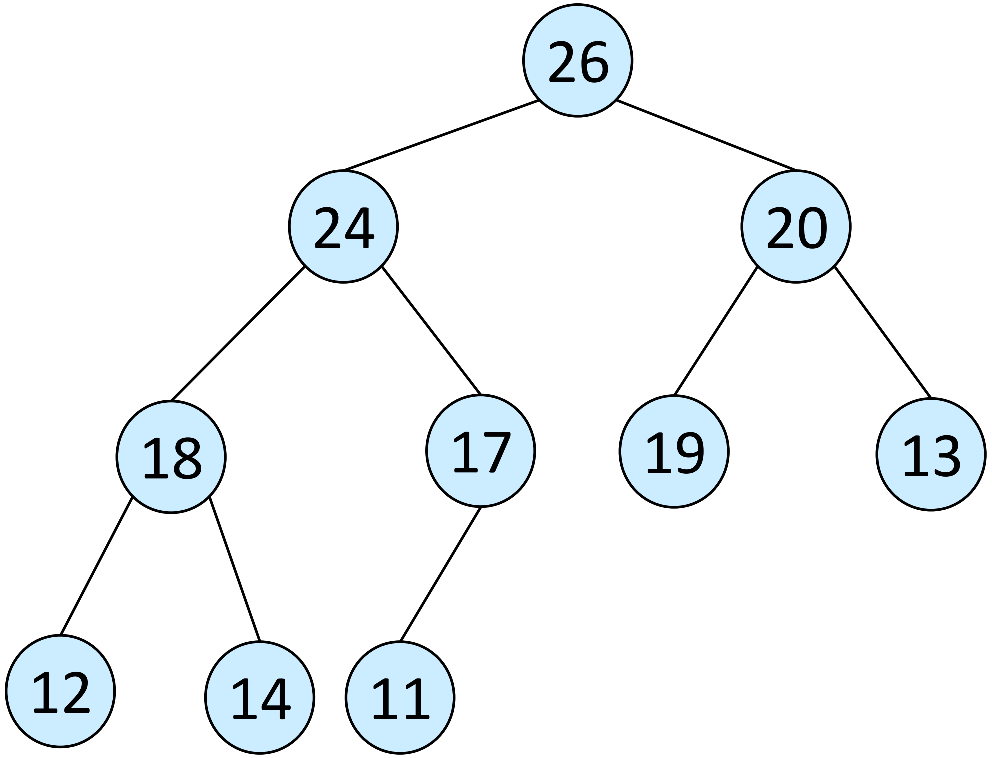

Algorithm Analysis

- In-place

- Not Stable

- Build-Max-Heap takes $O(n)$ and each of the n-1 calls to Max-Heapify takes time $O(\log n)$.

- Therefore, $T(n) = O(n \log n)$

<u>*HeapSort(A)*</u>

1. Build-Max-Heap(A)
2. **for** i = *length*[A] **downto** 2
3.   **do** exchange $A[1] \leftrightarrow A[i]$
4.   *heap-size*[A] = *heap-size*[A] – 1
5.   *MaxHeapify*(A, 1)

Heap Procedures for Sorting

- MaxHeapify $O(\log n)$
- BuildMaxHeap $O(n)$
- HeapSort $O(n \log n)$

Priority Queue

- Popular & important application of heaps.
- Max and min priority queues.
- Maintains a *dynamic* set *S* of elements.
- Each set element has a *key* – an associated value.
- Goal is to support insertion and extraction efficiently.

Priority Queue

- Popular & important application of heaps.
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- Maintains a *dynamic* set *S* of elements.
- Each set element has a *key* – an associated value.
- Goal is to support insertion and extraction efficiently.
- Applications:
  - Ready list of processes in operating systems by their priorities – the list is highly dynamic
  - In event-driven simulators to maintain the list of events to be simulated in order of their time of occurrence.

Basic Operations

- Operations on a max-priority queue:
  - Insert(S, x) - inserts the element x into the set S
    - $S \leftarrow S \cup \{x\}$.
  - Maximum(S) - returns the element of S with the largest key.
  - Extract-Max(S) - removes and returns the element of S with the largest key.
  - Increase-Key(S, x, k) – increases the value of element x's key to the new value k.

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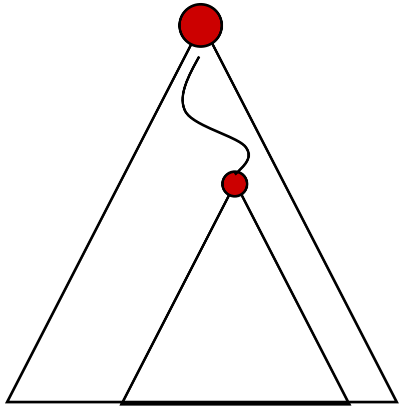
  - Extract-Max(S) - removes and returns the element of S with the largest key.
  - Increase-Key(S, x, k) – increases the value of element x's key to the new value k.
- Min-priority queue supports Insert, Minimum, Extract-Min, and Decrease-Key.
- Heap gives a good compromise between fast insertion but slow extraction and vice versa.

Heap Property (Max and Min)

- Max-Heap
  - For every node excluding the root, value is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
- In any subtree, no values are larger than the value stored at subtree root.



- Min-Heap
  - For every node excluding the root, value is at least that of its parent: $A[parent[i]] \leq A[i]$
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Heap-Extract-Max(A)

Implements the Extract-Max operation.

<u>*Heap-Extract-Max*(*A*)</u>

1. if *heap-size*[*A*] < 1
2. then error "heap underflow"
3. *max* = *A*[1]
4. *A*[1] = *A*[*heap-size*[*A*]]
5. *heap-size*[*A*] = *heap-size*[*A*] - 1
6. MaxHeapify(*A*, 1)
7. return max

Running time :
Dominated by the running time of MaxHeapify
$= O(\log n)$

Heap-Insert(A, key)

Heap-Insert(A, key)

1. $heap\text{-}size[A] = heap\text{-}size[A] + 1$
2. $i = heap\text{-}size[A]$
4. **while** $i > 1$ **and** $A[\text{Parent}(i)] < key$
5. **do** $A[i] = A[\text{Parent}(i)]$
6. $i = \text{Parent}(i)$
7. $A[i] = key$

Running time is $O(\log n)$

The path traced from the new leaf to the root has length $O(\log n)$

Heap-Increase-Key(A, i, key)

```
Heap-Increase-Key(A, i, key)
If key < A[i]
    then error "new key is smaller than the current key"
A[i] = key
while i > 1 and  A[Parent[i]] < A[i]
        do exchange A[i] ↔ A[Parent[i]]
            i = Parent[i]
```

```
Heap-Insert(A, key)
heap-size[A] = heap-size[A] + 1
A[heap-size[A]] = –∞
Heap-Increase-Key(A, heap-size[A], key)
```

Examples

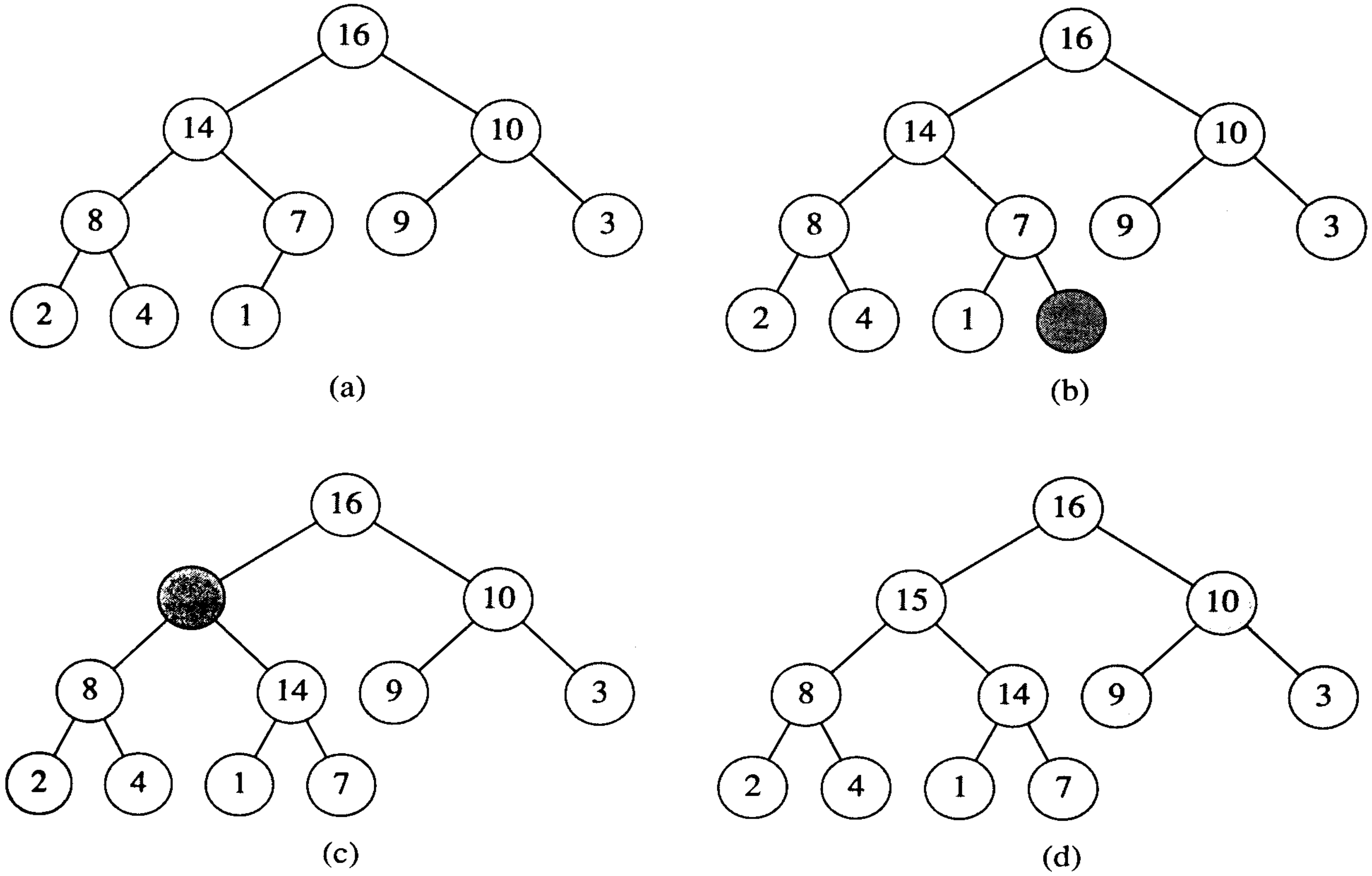

Figure 7.5 The operation of HEAP-INSERT. **(a)** The heap of Figure 7.4(a) before we insert a node with key 15. **(b)** A new leaf is added to the tree. **(c)** Values on the path from the new leaf to the root are copied down until a place for the key 15 is found. **(d)** The key 15 is inserted.

Acknowledgement

- University of North Carolina at Chapel Hill

Thank You