Problem Set-3

1. Which of the following maps are linear.

   (a) $T : \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by $T(x) = (x, 2x, 3x)$.
   (b) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x, y) = (2x + 3y, 3x - 4y)$.
   (c) $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ defined by $T(p(x)) = xp(x) + p(1)$.
   (d) $T : C[0, 1] \longrightarrow \mathbb{R}^2$ defined by $T(f) = (f(0), f(1))$.

2. Determine whether there exists a linear map in the following cases. If it exists, give the general formula.

   (a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $T(1, 2) = (3, 0)$ and $T(2, 1) = (1, 2)$.
   (b) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $T(0, 1, 2) = (3, 1, 2)$ and $T(1, 1, 1) = (2, 2, 2)$.
   (c) $T : \mathbb{P}_4(\mathbb{R}) \longrightarrow \mathbb{P}_3(\mathbb{R})$ such that $T(1 + x) = 1$, $T(x) = 3$ and $T(x^2) = 4$.
   (d) $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ such that $T(i, i) = (1 + i, 1)$.

3. Determine a linear map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, which maps all the vectors on the line $x + y = 0$ onto themselves.

4. Determine the Range and Kernel of the following linear maps. Also find the Rank and Nullity of T, if it exists

   (a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$.
   (b) $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ defined by $T(p(x)) = p''(x) - 2p(x)$.
   (c) $T : C(0, 1) \longrightarrow C(0, 1)$ defined by $T(f)(x) = f(x)\, sinx$.
   (d) $T : C^1[0, 1] \longrightarrow C[0, 1$ defined by $T(p(x)) = p'(x)\, e^x$, where $C^1[0, 1]$ is the set of all continuous function from $[0, 1]$ to $\mathbb{R}$ which are differentiable on $(0, 1)$. (Hint: Use Fundamental Theorem of Calculus)

5. Find a linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the set of all vectors (x_1, x_2, x_3) satisfying the equation $4x_1 - 3x_2 + x_3 = 0$ is the Kernel of T.

6. Let W be a subspace of $\mathbb{R}^4$ defined by $W = \{(x_1, x_2, x_3, x_4) \,|\, x_2 = 0\}$. Prove, by exhibiting an isomorphism, that $W \cong \mathbb{R}^3$.

7. True or false? Justify your answer.

   (a) There exist two isomorphism from $\mathbb{P}_2(\mathbb{R})$ to $\mathbb{R}^3$.
   (b) In $\mathbb{R}^2$ all nontrivial subspaces are isomorphic.

8. Let T be a linear map on a finite dimensional vector space V. Then prove that

   (a) $Range(T) \cap Ker(T) = \{0\}$ if and only if $T^2x = 0 \Rightarrow Tx = 0$.
   (b) If $Rank(T^2) = Rank(T)$, then $Range(T) \cap Ker(T) = \{0\}$.

9. Let $T : V \longrightarrow V$ (V is a finite dimensional vector space) be a linear map. If $Range(T) \cap Ker(T) = \{0\}$, then $V = Range(T) \oplus Ker(T)$.

10. Let $T : U \longrightarrow V$ and $S : V \longrightarrow W$ be two linear maps. Then prove that

   (a) If T is onto, then $Rank(ST) = Rank(S)$.

(b) If S is one-one, then $Rank(ST) = Rank(T)$.

11. Let T be a linear map on $\mathbb{R}^3$, defined by $T(x_1, x_2, x_3) = (3x_1, x_2, x_3)$. Show that T is invertible. Also find T^{-1}.

12. Let U and V be finite dimensional vector spaces over the field F such that $dim(U) = dim(V) = p$. If T is a linear map from U into V, the following are equivalent.

    (a) T is non-singular.
    (b) T is one one.
    (c) T transforms linearly independent subsets of U into linearly independent subsets of V.
    (d) T transforms every basis for U into a basis for V.
    (e) T is onto.
    (f) $Rank(T) = p$.
    (g) $Nullity(T) = 0$.
    (h) T^{-1} exists.

13. If a linear map T on V satisfies the condition $T^2 + I = T$, then prove that T^{-1} exists.

14. Consider non-zero finite dimensional real vector spaces V_1, V_2, V_3, V_4 and linear transformations $\phi_1 : V_1 \longrightarrow V_2, \phi_2 : V_2 \longrightarrow V_3$ and $\phi_3 : V_3 \longrightarrow V_4$ such that Ker $\phi_1 = \{0\}$, Range $\phi_1 =$ Ker ϕ_2, Range $\phi_2 =$ Ker ϕ_3, Range $\phi_3 = V_4$. Then find the value of $\sum_{i=1}^{4}(-1)^i \dim V_i$.

15. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear map defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, 12x_1 + x_2, -x_1 + 2x_2 + 4x_3)$.

    (a) Find the matrix M of T relative to the basis $B = (1, 0, 1), (-1, 2, 1), (2, 1, 1)$.
    (b) Find the matrix N of T relative to the standard basis of $\mathbb{R}^3$.
    (c) Find a non singular matrix P such that $N = PMP^{-1}$.

16. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear map defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$. Let $B = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$ be an ordered basis of $\mathbb{R}^3$.

    (a) Find the matrix M of T relative to the pair B and standard basis of $\mathbb{R}^2$.
    (b) Find the matrix N of T relative to the standard basis of $\mathbb{R}^3$ standard basis of $\mathbb{R}^2$.
    (c) Find a non singular 2×2 matrix P and a non singular 3×3 matrix Q such that $N = PMQ$.

17. Let A be an $n \times n$ invertible matrix.

    (a) Let B be an $n \times k$ matrix. Show that $\text{Rank}(AB) = \text{Rank}(B)$.
    (b) Let B be an $n \times n$ invertible matrix. Then show that $\text{Rank}(AB) = \text{Rank}(A) = Rank(B)$.

18. Let $A \in M_{m \times n}(\mathbb{R})$. Then $\text{Rank}(AA^t) = \text{Rank}(A)$.

19. Let A be an $m \times n$ matrix and R be the RRE form of A. Then show that $\text{Rank}(A) = \text{Rank}(R)$.

20. Let $V = M_2(\mathbb{R})$ and $W = P_3(\mathbb{R})$ be two vector spaces over $\mathbb{R}$. Define the linear transformation $T : V \to W$ by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2a + (b - d)x - (a + c)x^2 + (a + b - c - d)x^3.$$

(a) Find the matrix representation of T with respect to the standard ordered bases.

(b) Find a basis for the range of T and a basis for the null space of T.

(c) Find Rank(T) and Nullity(T).