Problem Set 2

1. Let $A \in M_n(\mathbb{R})$ and c_{ij} be the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and the j-th column. Then (adj A) denotes the adjoint matrix of A which is given by $(\text{adj}A)_{ij} = (-1)^{i+j} c_{ji}$. Then

   (a) $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} c_{ij} \ \forall \, 1 \leq i \leq n.$

   (b) $(\text{adj}A)A = (\det A) I_n$.

   (c) if A is invertible, then $A^{-1} = \frac{\text{adj } A}{\det A}$.

   (d) if A is an invertible matrix with integer entries, then A^{-1} has integer entries if and only if $\det(A) = \pm 1$.

2. Evaluate determinant of the following $n \times n$ matrix

   (a) $$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

   (b) $$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \cdots & n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n & n & n & \cdots & n \end{pmatrix}.$$

3. For which of the following, V forms a vector space over $\mathbb{R}$? If not, then justify your answer.

   (a) $V = \mathbb{R}^2$ with addition : $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and scalar multiplication : $\alpha(x, y) = (\alpha x, y)$.

   (b) $V = \mathbb{R}^2$ with addition : $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 - y_2)$ and scalar multiplication : $\alpha(x, y) = (\alpha x, \alpha y)$.

   (c) $V = \{f(x) \in \mathbb{R}[x] \mid \deg f(x) > n\}$ with usual addition and scalar multiplication.

   (d) $V = \{c_1 e^x + c_2 e^{3x} \mid c_1, \ c_2 \in \mathbb{R}\}$ with usual addition and scalar multiplication.

4. (a) Let $\mathbb{H} = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}\}$. Show that $\mathbb{H}$ is a vector space (under the usual matrix addition and scalar multiplication) over $\mathbb{R}$.

   (b) Is $\mathbb{H}$ a vector space over $\mathbb{C}$?

5. Determine whether W is a subspace of $V(\mathbb{F})$. Justify your answer.

   (a) $W = \{\bar{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$ and $V = \mathbb{R}^n$ and $\mathbb{F} = \mathbb{R}$.

(b) $W = \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R}\}$ and $V = \mathbb{P}_2(\mathbb{R})$ and $\mathbb{F} = \mathbb{R}$.

(c) $W = \{A \in V \mid |A| = 0\}$ and $V = M_n(\mathbb{R})$ and $\mathbb{F} = \mathbb{R}$.

(d) $W = \{p(x) \in V \mid p(0) = 5\}$ and $V = \mathbb{P}_2(\mathbb{R})$ and $\mathbb{F} = \mathbb{R}$.

(e) $W = \{A \in V \mid |A| = 1\}$ and $V = M_n(\mathbb{R})$ and $\mathbb{F} = \mathbb{R}$.

(f) $W = \{A \in V \mid |A| \neq 0\}$ and $V = M_n(\mathbb{R})$ and $\mathbb{F} = \mathbb{R}$.

6. Discuss the linear dependence/linear independence of the following sets:

   (a) $S = \{(1,0,0), (1,1,0), (1,1,1)\}$ of $\mathbb{R}^3(\mathbb{R})$.

   (b) $S = \{(1,i,0), (1,0,1), (i+2,-1,2)\}$ of $\mathbb{C}^3(\mathbb{C})$.

   (c) $S = \{(1,i,0), (1,0,1), (i+2,-1,2)\}$ of $\mathbb{C}^3(\mathbb{R})$

7. Show that $C(X)$ is a vector space over $\mathbb{R}$ with addition : $(f+g)(x) = f(x) + g(x)$ and scalar multiplication : $(\alpha f)(x) = \alpha f(x)$. Also show that the set $S = \{\sin x, \sin 2x, \ldots, \sin nx\}$ is a linearly independent subset of $C([-\pi, \pi])$.

8. Show that $X = \{(1+i, 1-i), (1-i, 1+i), (2, i), (3, 2i)\}$is linearly independent in $\mathbb{C}^2(\mathbb{R})$. Write $(a+ib, c+id)$ as an $\mathbb{R}$-linear combination of vectors belonging to X.

9. Suppose $V = \mathbb{C}^2(\mathbb{C})$. Then

   (a) Show that $\{(1+i, 2), (2, 1)\}$ is linearly independent.

   (b) Show that $\{(1,2), (0,i), (i, 1-i)\}$ is linearly dependent.

   (c) Show that every ordered pair in $\mathbb{C}^2$ can be written as a linear combination of $v_1 = (1,2), v_2 = (0,i), v_3 = (i, 1-i)$ over $\mathbb{C}$ in more than one ways.

10. Prove that the following set forms a subspace of $M_n(\mathbb{R})$, and find a basis and the dimension of each subspace.

    (a) The set of all diagonal matrices.

    (b) The set of all symmetric matrices.

    (c) The set of all skew-symmetric matrices.

11. For which real number x, the vectors: $(x,1,1,1), (1,x,1,1), (1,1,x,1), (1,1,1,x)$ do not form a basis of $\mathbb{R}^4(\mathbb{R})$? For each of the values of x that you find, what is the dimension of the subspace of $\mathbb{R}^4$ that they span?

12. Let v_1, v_2, v_3 and v_4 be linearly independent. Justify if the following vectors are linear independent or not?

    (a) $v_1 + v_2, v_2 + v_3, v_3 - v_4$ and $v_4 - v_1$.

    (b) $v_1 - v_2, v_2 - v_3, v_3 - v_4$ and $v_4 - v_1$.

13. Find the dimension and a basis of the set of solutions of the system of equations

    (a) $x_1 + 2x_2 = 0, x_2 - x_3 = 0, x_1 + x_2 + x_3 = 0$.

    (b) $2x_1 + 4x_2 - 5x_3 + 3x_4 = 0, 3x_1 + 6x_2 - 7x_3 + 4x_4 = 0, 5x_1 + 10x_2 - 11x_3 + 6x_4 = 0$.

14. Prove that the following set of vectors in $\mathbb{R}^n(\mathbb{R})$ forms a subspace and find a basis and the dimension of each

    (a) all n-vectors whose odd entries are equal

    (b) all n-vectors whose odd entries are zero

    (c) all n-vectors whose first and the last co-ordinates are equal.

15. Let V be a vector space of dimension 100, W_1 and W_2 are two subspaces of V of dimensions 60 and 63. What is the minimum dimension and the maximum dimension of $W_1 \cap W_2$.

16. Let V be a vector space of dimension n. Let W_1 be a subspace of dimension $n-1$ and W_2 be a subspace of dimension r such that W_2 is not contained in W_1. Then find the dimension of $W_1 \cap W_2$.

17. Let V be a vector space of dimension n. Let W_1 and W_2 be distinct subspaces of dimension $n-1$ each. Then find the dimension of $W_1 \cap W_2$.

18. Let W be a subspace of V.

    (a) Prove that there is a subspace U of V such that $U + W = V$ and $U \cap W = \{0\}$.

    (b) Prove that there is no subspace U such that $U \cap W = \{0\}$ and that dim W + dim $U >$ dim V.

19. Find the coordinates of the polynomial $x^5 - x^4 + x^3 - x^2 + x + 1$ with respect to the following ordered basis

    (a) $\{1 + x^3, x + x^3, x^2 + x^3, x^3, x^4 + x^3, x^5 + x^3\}$ of $\mathbb{P}_5(\mathbb{R})$.

    (b) $\{1, x + 1, x^2 + 1, x^3 + 1, x^4 + 1, x^5 + 1\}$ of $\mathbb{P}_5(\mathbb{R})$.