Notations:

$\mathbb{N}$ - the set of all natural numbers
$\mathbb{Z}$ - the set of all integers
$\mathbb{Q}$ - the set of all rational numbers
$\mathbb{R}$ - the set of all real numbers
$\mathbb{C}$ - the set of all complex numbers
$\bar{a}$ - complex conjugate of $a \in \mathbb{C}$
A^t - the transpose of the matrix A
$\bar{A}$ - the complex conjugate of the matrix A
$|A|$ - the determinant of the square matrix A
$M_{m\times n}(S)$ - the set of all $m \times n$ matrices with entries in the set S, where $S = \mathbb{R}$ or $\mathbb{C}$
$M_n(S)$ - the set of all $n \times n$ matrices with entries in the set S, where $S = \mathbb{R}$ or $\mathbb{C}$
$GL_n(S) = \{A \in M_n(S) \mid |A| \neq 0\}$ - the set of all $n \times n$ invertible matrices with entries in the set S, where $S = \mathbb{R}$ or $\mathbb{C}$. This is called general linear group.
$SL_n(S) = \{A \in M_n(S) \mid |A| = 1\}$ - the set of all $n \times n$ invertible matrices with determinant 1, where $S = \mathbb{R}$ or $\mathbb{C}$. This is called special general linear group.
$S_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^t = A\}$ - the set of all $n \times n$ symmetric matrices
$SS_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^t = -A\}$ - the set of all $n \times n$ skew-symmetric matrices
$H_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \bar{A}^t = A\}$ - the set of all $n \times n$ hermitian matrices
$SH_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \bar{A}^t = -A\}$ - the set of all $n \times n$ skew-hermitian matrices
$S[x]$ - the set of all polynomials with coefficient from the S
$\mathbb{P}_n(S)$ - the set of all polynomials with coefficient from the S of degree $\leq n$
$C(X)$ - the set of all continuous functions from the set X to $\mathbb{R}$ i.e. the set of all real valued continuous functions on X

Problem Set- I

1. Show that $(M_{m\times n}(\mathbb{R}), +)$ is a group, where '$+$' denotes matrix addition. What can you say about the followings:

   (a) $(M_n(\mathbb{R}), \cdot)$, where '$\cdot$' denotes matrix multiplication.

   (b) $(GL_n(\mathbb{R}), \cdot)$, where $GL_n(\mathbb{R})$ is the collection of $n \times n$ invertible matrices and '$\cdot$' denotes matrix multiplication.

   (c) $(SL_n(\mathbb{R}), \cdot)$, where $SL_n(\mathbb{R})$ is the collection of $n \times n$ matrices with determinant 1 and '$\cdot$' denotes matrix multiplication.

2. Let $A, B \in M_n(\mathbb{R})$.

   (a) Show by an example that if $AB \neq BA$ then $(A+B)^2 = A^2 + 2AB + B^2$ need not hold.

   (b) If $AB = BA$ then show that $(A+B)^m = \Sigma_{i=0}^{m} \binom{m}{i} A^{m-i}B^i$.

3. Find all elements of S_3 (the set of all permutations of the set $\{1, 2, 3\}$) and determine which permutations are odd.

4. Let $\sigma \in S_5$ be given by $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$.
   (a) Find sign of σ and σ^{-1}.
   (b) Find $\sigma \circ \tau$ and $\tau \circ \sigma$, where $\tau = (1\,2\,3\,4\,5)$.

5. Denote an elementary row operation by ρ. If $A \in M_{m\times n}(\mathbb{R})$ then $\rho(A) = \rho(I) \cdot A$, where I is the $m \times m$ identity matrix.

6. Decide if they are row-equivalent: (i) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & -1 \\ 5 & -1 & 5 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 4 \end{pmatrix}$

7. Which of the following matrices are elementary?
   (i) $\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

8. Suppose $Ax = b$ and $Cx = b$ have same solutions for every b. Is it true that $A = C$?

9. Determine whether the given matrix is in row reduced echelon form, row echelon form, or neither
   (i) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

10. Find a matrix in row echelon form that is equivalent to the given matrix. Give two possible answers in each.

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 3 & 0 & 2 \\ 5 & 7 & -9 & 2 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 3 \\ 4 & 7 & -8 \end{pmatrix}$ (iv) $\begin{pmatrix} 0 & 2 & 4 & 3 & 0 \\ 0 & 5 & 10 & 15/2 & 0 \\ 0 & 1 & 2 & 3/2 & 4 \\ 0 & 2 & 4 & 3 & 2 \end{pmatrix}$

11. Describe explicitly all 2×2, 3×2 and 3×3 row reduced echelon matrices.

12. Find the inverse of the following matrices using Gauss - Jordon method.

(i) $\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 5 \\ 2 & 5 & 11 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & -3 & 2 \\ 2 & 0 & 0 \\ 1 & 4 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 2 \end{pmatrix}$

13. Find non-singular matrices P and Q so that PAQ is in normal form for the following matrices:

(i) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 2 & 3 \\ 0 & -1 & -3 & -2 \end{pmatrix}$

14. Let B be an invertible matrix. Then, prove that the system $Ax = b$ and $BAx = Bb$ are equivalent.

15. Let $\lambda = 2, 3, 5$. Find the all solutions of $AX = \lambda X$, where $A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$.

16. For an $n \times n$ matrix A, show that the following statements are equivalent.

    (a) A is invertible.

    (b) The homogeneous system $AX = 0$ has only the trivial solution.

    (c) The system $AX = Y$ has a solution for each $n \times 1$ matrix Y.

17. An $n \times n$ matrix $A = (a_{ij})$ is called upper-triangular if $a_{ij} = 0$ for $i > j$ i.e. if every entry below the main diagonal is 0. Prove that an upper-triangular matrix is invertible if and only if every entry on its main diagonal is different from 0.

18. Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Then prove that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

19. Find all the solutions of following system of linear equations using Gauss-Jordon elimination method, if exists:

    (a) $x_1 - x_2 + 2x_3 = 1,\ 2x_1 + 2x_3 = 1,\ x_1 - 3x_2 + 4x_3 = 2$

    (b) $x_1 - 2x_2 + x_3 + 2x_4 = 1,\ x_1 + x_2 - x_3 + x_4 = 2,\ x_1 + 7x_2 - 5x_3 - x_4 = 3$

    (c) $x_1 + 3x_2 - 2x_3 = 3,\ 2x_1 + 6x_2 - 2x_3 + 4x_4 = 18,\ x_2 + x_3 + 3x_4 = 10$

    (d) $5x_1 + 2x_2 + 7x_3 = 4,\ 3x_1 + 26x_2 + 2x_3 = 9,\ 7x_1 + 2x_2 + 10x_3 = 5$

    (e) $x_2 + 4x_3 + 2x_4 = -5,\ x_1 + 3x_2 + 5x_3 + x_4 = -2,\ 3x_1 + 7x_2 + 7x_3 - x_4 = 6$

(f) $2x_1 - 2x_2 + 4x_4 = 2, \ -x_1 + 3x_3 + x_4 = 6, \ 6x_1 - 6x_2 + x_3 + 8x_4 = 3$

20. Find the condition on a so that the linear system

$$x + 2y - 3z = 2, \ 2x + 6y - 11z = 4, \ x - 2y + 7z = a$$

has a solution. Find the solution set.