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Research Article

Pell Equations and $\mathscr{F}_{p^l}$-Continued Fractions

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In this note, the solvability of the Pell equation, $X^2 - DY^2 = 1$, is discussed over $\mathbb{Z} \times p^l\mathbb{Z}$. In particular, we show that this equation is solvable over $\mathbb{Z} \times p^l\mathbb{Z}$ for each prime p and natural number l. Moreover, we show that solutions to the Pell equation over $\mathbb{Z} \times p^l\mathbb{Z}$ are completely determined by the $\mathscr{F}_{p^l}$-continued fraction expansion of $\sqrt{D}$.

1. Introduction

A Diophantine equation of the form $X^2 - DY^2 = 1$ is known as the Pell equation, where D is a nonsquare positive integer. Finding solutions to the Pell equation has always been an interesting problem.

In this note, we look for solutions to the Pell equation, $X^2 - DY^2 = 1$, in $\mathbb{Z} \times p^l\mathbb{Z}$, where p is an odd prime and $l \in \mathbb{N}$. The problem has been discussed for $p = 2$ by the authors in [1]. It is well known that $X^2 - DY^2 = 1$ is always solvable in $\mathbb{Z} \times \mathbb{Z}$. Suppose (X_0, Y_0) is a solution of $X^2 - DY^2 = 1$ in $\mathbb{Z} \times \mathbb{Z}$. Then, (X_1, Y_1) is obtained by comparing $(X_1 + \sqrt{D}Y_1) = (X_0 + \sqrt{D}Y_0)^{2^l}$, which is a solution of Pell equation in $\mathbb{Z} \times 2^l\mathbb{Z}$. Given a solution $(X_1, Y_1) \in \mathbb{Z} \times 2^l\mathbb{Z}$, one can find infinitely many solutions, $(X_{n+1}, Y_{n+1}) \in \mathbb{Z} \times 2^l\mathbb{Z}$ for $n \geq 0$, by the following equation:

$$\left(X_{n+1} + \sqrt{D}Y_{n+1}\right) = \left(X_1 + \sqrt{D}Y_1\right)^{2^n}. \tag{1}$$

But this idea does not work for an odd prime. For instance, let $D = 5$, then $(X_0, Y_0) = (9, 4)$ and any solution of the equation can be determined by computing $(X_0 + \sqrt{D}Y_0)^i$, where $i \geq 1$. Putting $i = 3$, we get a solution $(2889, 1292)$, which does not belong to $\mathbb{Z} \times 3\mathbb{Z}$. One can see that a solution obtained by computing $(X_0 + \sqrt{D}Y_0)^{p^l}$ does not belong to $\mathbb{Z} \times p^l\mathbb{Z}$, where (X_0, Y_0) is the minimal solution of $X^2 - DY^2 = 1$. Thus, we raise a question to discuss the solvability of $X^2 - DY^2 = 1$ in $\mathbb{Z} \times p^l\mathbb{Z}$ when p is an odd prime.

In 2016, Luca et al. proposed a potentially interesting problem related to the Pell equation. Suppose Z is a subset of natural numbers. The problem can be stated as discussing the solvability of the Pell equation over a favorable set of $Z \times Z$ and finding D for which there are more than one solution of the required form. A lot of development can be seen in this direction [2–9]. One can consider a similar problem with the second coordinate of the Pell equation. Here, we discuss this problem when $Z = \mathscr{X}_{p^l}$, where

$$\mathscr{X}_{p^l} = \left\{ \frac{r}{p^l s} \colon r, s \in \mathbb{Z}, s > 0, (r, ps) = 1 \right\} \cup \{\infty\}. \tag{2}$$

Moreover, a solution to the Pell equation with the given restriction is related to certain continued fractions. $\mathscr{F}_{p^l}$-continued fractions and their properties have been studied by Kushwaha et al. in [10–13]. A finite continued fraction of the form

$$\frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1+}\,\frac{\varepsilon_2}{a_2+}\cdots\frac{\varepsilon_n}{a_n}\,(n \geq 0) \tag{3}$$

or an infinite continued fraction of the form

$$\frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1+}\,\frac{\varepsilon_2}{a_2+}\cdots\frac{\varepsilon_n}{a_n+}\cdots, \tag{4}$$

where b is an odd integer, $a_1, a_2, \ldots$ are positive integers coprime to p, and $\varepsilon_1, \varepsilon_2, \ldots \in \{\pm 1\}$, with certain conditions on a_i and ε_i is called an $\mathscr{F}_{p^l}$*-continued fraction*. Every irrational number has a unique infinite $\mathscr{F}_{p^l}$-continued fraction expansion. The expression

$$\frac{P_i}{Q_i} = \frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1+}\,\frac{\varepsilon_2}{a_2+}\cdots\frac{\varepsilon_i}{a_i} \tag{5}$$

for $i \geq 0$ is called the i-th $\mathscr{F}_{p^l}$-convergent which belongs to $\mathscr{X}_{p^l}$, where

$$\mathscr{X}_{p^l} = \left\{\frac{r}{p^l s}\colon r, s \in \mathbb{Z}, s > 0, (r, ps) = 1\right\} \cup \{\infty\}. \tag{6}$$

The $\mathscr{F}_{p^l}$-continued fractions also characterize best approximations of a real number by elements of $\mathscr{X}_{p^l}$, these approximations are defined in the following way.

A rational number $r/s \in \mathscr{X}_{p^l}$ is called the best approximation of α by an element of $\mathscr{X}_{p^l}$, if for every $r'/s' \in \mathscr{X}_{p^l}$ different from r/s with $0 < s' \leq s$, we have $|s\alpha - r| < |s'\alpha - r'|$.

Note that a solution $(P, Q) \in \mathbb{Z} \times p^l\mathbb{Z}$ to $X^2 - DY^2 = \pm 1$ ensures that $P/Q \in \mathscr{X}_{p^l}$. Thus, we raise the question to solve the Pell equation in $\mathbb{Z} \times p^l\mathbb{Z}$ by using $\mathscr{F}_{p^l}$-continued fractions. The organization of this article is as follows: Section 2 recalls the known properties of $\mathscr{F}_{p^l}$-continued fractions. We derive certain results which we will use to prove our main results. Section 3 deals with the question of the periodicity of an $\mathscr{F}_{p^l}$-continued fraction. In particular, we show that an irrational number has a periodic $\mathscr{F}_{p^l}$-continued fraction if and only if it is a quadratic surd. The notion of pure periodicity of $\mathscr{F}_{p^l}$-continued fractions is introduced, and related results are proved. In Section 4, we achieve our main results related to the solvability of Pell's equation in $\mathbb{Z} \times p^l\mathbb{Z}$. We conclude this section by adding a remark on the contribution of our results to algebraic number theory.

2. Preliminaries

We summarize the basic results of $\mathscr{F}_{p^l}$-continued fractions (for more details refer to [11, 12]). For basic properties of regular continued fractions and semi-regular continued fractions we refer to [14, 15]. Furthermore, we derive certain results related to $\mathscr{F}_{p^l}$-continued fractions, which we will use in the forthcoming sections.

Definition 1. Suppose p is a prime and $l \in \mathbb{N}$. A finite continued fraction of the form

$$\frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1+}\,\frac{\varepsilon_2}{a_2+}\cdots\frac{\varepsilon_n}{a_n}\,(n \geq 0) \tag{7}$$

or an infinite continued fraction of the form

$$\frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1+}\,\frac{\varepsilon_2}{a_2+}\cdots\frac{\varepsilon_n}{a_n+}\cdots, \tag{8}$$

where b is an integer coprime to p, $a_1, a_2, \ldots$ are positive integers, and $\varepsilon_1, \varepsilon_2, \ldots \in \{\pm 1\}$, such that $a_i + \varepsilon_{i+1} \geq 1$, $a_i + \varepsilon_i \geq 1$, and $\gcd(P_i, Q_i) = 1$ with $P_i = a_i P_{i-1} + \varepsilon_i P_{i-2}$, $Q_i = a_i Q_{i-1} + \varepsilon_i Q_{i-2}$, $(P_{-1}, Q_{-1}) = (1, 0)$, and $(P_0, Q_0) = (b, p^l)$ is called an $\mathscr{F}_{p^l}$-continued fraction.

Given an $\mathscr{F}_{p^l}$-continued fraction

$$\frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1+}\,\frac{\varepsilon_2}{a_2+}\,\frac{\varepsilon_3}{a_3+}\cdots\frac{\varepsilon_n}{a_n+}\cdots, \tag{9}$$

the following continued fraction

$$\frac{\varepsilon_i}{a_i+}\,\frac{\varepsilon_{i+1}}{a_{i+1}+}\cdots\frac{\varepsilon_n}{a_n+}\cdots \tag{10}$$

is called the fin at the i-th stage of the $\mathscr{F}_{p^l}$-continued fraction for $i \geq 1$. Here, we record certain propositions describing properties of $\mathscr{F}_{p^l}$-continued fractions.

Theorem 1 (see [11], Theorem 3.2). *Suppose $x = 1/0+ p^l/b+\varepsilon_1/a_1+\varepsilon_2/a_2+\varepsilon_3/a_3+\cdots$ is an $\mathscr{F}_{p^l}$-continued fraction with the sequence of convergence $\{P_i/Q_i\}_{i\geq -1}$. Let y_i be the i-th fin of the continued fraction. Then,*

(1) $P_i Q_{i-1} - Q_i P_{i-1} = \pm p^l$
(2) $i \geq 1$, $a_i \equiv -\varepsilon_i P_{i-2} P_{i-1}^{-1} \bmod p$
(3) The sequence $\{Q_i\}_{i\geq -1}$ is strictly increasing
(4) $P_i/Q_i \neq P_j/Q_j$ for $i \neq j$
(5) For $i \geq 1$, $|y_i| \leq 1$
(6) For $n \geq 0$, $x = x_{n+1}P_n + \varepsilon_{n+1}P_{n-1}/x_{n+1}Q_n + \varepsilon_{n+1}Q_{n-1}$, where $x_i = 1/|y_i|, i \geq 0$

Definition 2. Suppose $x \in \mathscr{X}_{p^l}$. An $\mathscr{F}_{p^l}$-continued fraction of x not ending with 1/1 is said to be an $\mathscr{F}_{p^l}$-continued fraction with a maximum +1 if it has the maximum number of positive partial numerators excluding ε_1, the first partial numerator, among all its $\mathscr{F}_{p^l}$-continued fraction expansions.

An infinite $\mathscr{F}_{p^l}$-continued fraction

$$\frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1+}\,\frac{\varepsilon_2}{a_2+}\cdots\frac{\varepsilon_n}{a_n+}\cdots \tag{11}$$

is said to be an $\mathscr{F}_{p^l}$-continued fraction with maximum +1 if

$$\frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1+}\,\frac{\varepsilon_2}{a_2+}\cdots\frac{\varepsilon_i}{a_i} \tag{12}$$

is an $\mathscr{F}_{p^l}$-continued fraction with a maximum +1 of the i-th convergent unless $(\varepsilon_i, a_i) = (1, 1)$.

Theorem 2 (see [12], Theorem 3.6, Corollary 3.8). *Suppose x is an irrational number. Then,*

(1) There is a unique $\mathscr{F}_{p^l}$-continued fraction expansion of x with maximum +1.
(2) The $\mathscr{F}_{p^l}$-continued fraction expansion

$$\frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1+}\,\frac{\varepsilon_2}{a_2+}\cdots\frac{\varepsilon_n}{a_n}\cdots. \tag{13}$$

of x with maximum +1 is obtained as follows:

$$b = \begin{cases} \lfloor p^l x \rfloor, & \text{if } \left(\lfloor p^l x \rfloor + 1, p \right) \neq 1 \\ \lfloor p^l x \rfloor + 1, & \text{if } \left(\lfloor p^l x \rfloor, p \right) \neq 1 \\ \lfloor p^l x \rfloor, & \text{if } \left(\lfloor p^l x \rfloor, p \right) = 1 = \left(\lfloor p^l x \rfloor + 1, p \right) \text{and } x < \dfrac{\lfloor p^l x \rfloor}{p^l} \oplus \dfrac{\lfloor p^l x \rfloor + 1}{p^l}. \\ \lfloor p^l x \rfloor + 1, & \text{if } \left(\lfloor p^l x \rfloor, p \right) = 1 = \left(\lfloor p^l x \rfloor + 1, p \right) \text{and } x > \dfrac{\lfloor p^l x \rfloor}{p^l} \oplus \dfrac{\lfloor p^l x \rfloor + 1}{p^l} \end{cases} \tag{14}$$

Set $y_1 = p^l x - b$,

(a) $\varepsilon_i = \operatorname{sign}(y_i)$.

(b)

$$a_i = \begin{cases} \left\lceil \left(\dfrac{1}{|y_i|} - 1 \right) \right\rceil, & \text{if } \left\lceil \left(\dfrac{1}{|y_i|} - 1 \right) \right\rceil \not\equiv -\varepsilon_i p_{i-2} p_{i-1}^{-1} \operatorname{mod} p \\ \left\lfloor \left(\dfrac{1}{|y_i|} + 1 \right) \right\rfloor, & \text{otherwise.} \end{cases} \tag{15}$$

(c) $y_{i+1} = 1/|y_i| - a_i$.

Proposition 1 (see [12], Remark 2). *Suppose $x \in \mathbb{R}$ has an eventually constant $\mathcal{F}_{p^l}$-continued fraction. Then, $x \in \mathbb{Q}$ if and only if all but finitely many partial numerators are -1 and all but finitely many partial denominators are 2.*

Corollary 1. *Suppose α is an irrational number. Then, there are infinitely many $i \in \mathbb{N}$ such that $\varepsilon_i/a_i \neq -1/2$.*

In Section 1, we introduced the definition of the best approximation by an element in $\mathcal{X}_{p^l}$. The following theorem records the result on best approximation properties of $\mathcal{F}_{p^l}$-continued fractions.

Definition 3. A rational number $u/v \in \mathcal{X}_{p^l}$ is called a best approximation of $x \in \mathbb{R}$ by an element of $\mathcal{X}_{p^l}$, if for every $u'/v' \in \mathcal{X}_{p^l}$ different from u/v with $0 < v' \leq v$, we have $|vx - u| < |v'x - u'|$.

Theorem 3 (see [12], Theorem 4.9, 4.11). *Suppose α is an irrational number and $r/s \in \mathcal{X}_{p^l}$. Then, r/s is a best approximation of α by an element of $\mathcal{X}_{p^l}$ if and only if r/s is a convergent of the $\mathcal{F}_{p^l}$-continued fraction of α with maximum +1.*

Lemma 1. *Let α be a real number and P_i/Q_i be the sequence of convergence of the $\mathcal{F}_{p^l}$-continued fraction of α with maximum +1. Suppose P_n/Q_n is an $\mathcal{F}_{p^l}$-convergent of α with $\varepsilon_{n+1}/a_{n+1} \neq -1/2$. Then,*

$$\left| \alpha - \frac{P_n}{Q_n} \right| < \frac{p^l}{Q_n^2}. \tag{16}$$

Proof. Let y_i denote the i-th fin of the $\mathcal{F}_{p^l}$-continued fraction of α with maximum +1 and $x_i = 1/|y_i|$. By Theorem 1 (6),

$$\alpha = \frac{x_{n+1} P_n + \varepsilon_{n+1} P_{n-1}}{x_{n+1} Q_n + \varepsilon_{n+1} Q_{n-1}}. \tag{17}$$

Thus,

$$\begin{aligned} \left| \alpha - \frac{P_n}{Q_n} \right| &= \left| \frac{x_{n+1} P_n + \varepsilon_{n+1} P_{n-1}}{x_{n+1} Q_n + \varepsilon_{n+1} Q_{n-1}} - \frac{P_n}{Q_n} \right| \\ &= \frac{\left| \varepsilon_{n+1} \left(P_{n-1} Q_n - P_n Q_{n-1} \right) \right|}{\left| (x_{n+1} Q_n + \varepsilon_{n+1} Q_{n-1}) Q_n \right|} \\ &= \frac{p^l}{\left| x_{n+1} Q_n + \varepsilon_{n+1} Q_{n-1} \right| Q_n} \text{ (byTheorem 1 (1)).} \end{aligned} \tag{18}$$

If $\varepsilon_{n+1} = 1$, then $x_{n+1} Q_n + \varepsilon_{n+1} Q_{n-1} > q_n$. If $\varepsilon_{n+1} = -1$, we claim that $x_{n+1} \geq 2$. We know that $x_{n+1} \geq 1$. Let $1 \leq x_{n+1} < 2$. By Theorem 2, $a_{n+1} = \lceil (x_{n+1} - 1) \rceil$ or $a_{n+1} = \lfloor (x_{n+1} + 1) \rfloor$ so that $a_{n+1} = 1$ or 2. By definition of $\mathcal{F}_{p^l}$-continued fraction, $\varepsilon_{n+1} + a_{n+1} \geq 1$, since $\varepsilon_{n+1} = -1$, $a_{n+1} \neq 1$. By hypothesis, $\varepsilon_{n+1}/a_{n+1} \neq -1/2$ and hence $a_{n+1} \neq 2$. Therefore, $x_{n+1} \geq 2$ and hence $x_{n+1} Q_n + \varepsilon_{n+1} Q_{n-1} > q_n$. Thus, we get $|\alpha - P_n/Q_n| < p^l/Q_n^2$. □

Using Corollary 1, we have the following corollary of Lemma 1:

Corollary 2. *Suppose α is an irrational number. Then, there are infinitely many $r/s \in \mathscr{X}_{p^l}$, such that $|\alpha - r/s| < p^l/s^2$.*

3. Periodic $\mathscr{F}_{p^l}$-Continued Fractions

An $\mathscr{F}_{p^l}$-continued fraction is called periodic of period length $m \geq 1$ with an initial block of length $n \geq 1$, if $y_n \neq y_{n+r}$, for $r \geq 1$, but $y_{n+i} = y_{(n+km)+i}$, that is,

$$\varepsilon_{n+i} = \varepsilon_{(n+km)+i} \text{ and } \alpha_{n+i} = a_{(n+km)+i}, \tag{19}$$

for $1 \leq i \leq m$ and $k \geq 0$. The continued fraction with no initial block is called purely periodic. In this section, we discuss that a periodic $\mathscr{F}_{p^l}$-continued fraction reaches a quadratic surd and vice versa. Recall that a quadratic surd is a solution of a quadratic equation $Ax^2 + Bx + c = C$ with integer coefficients $A \neq 0, B$, and C such that the discriminant $D = B^2 - 4AC$ is not a perfect square. Here, we record an observation, which we will use further.

Lemma 2. *A real number α is a quadratic surd if and only if $u\alpha + v$ is a quadratic surd, where $0 \neq u \in \mathbb{Q}$ and $v \in \mathbb{Q}$.*

Lemma 3. *Suppose α is an irrational number and y_i is the i-th fin of the $\mathscr{F}_{p^l}$-continued fraction expansion of α with maximum $+1$. If $y_k = y_r$ for some k, r with $r > k$. Then, $y_{k+j} = y_{r+j}$, for each $j \geq 1$. In particular, the continued fraction is periodic.*

Proof. By Theorem 2, $y_{k+1} = 1/|y_k| - a_k$, where $a_k \equiv -\varepsilon_k P_{k-2} P_{k-1}^{-1} \bmod p$. Note that $\varepsilon_k = \varepsilon_r$ and $y_{k+1} \neq y_{r+1}$ if and only if $a_r = a_k \pm 1$. Here, we get a contradiction to the fact that $|y_{r+1}| < 1$. Thus, the statement is true for $j = 1$. Now, suppose for each $j > 1$, $y_{k+j} = y_{m+j}$. The proof is by induction. Using the fact that y_i is irrational for each $i \geq 1$ and applying the same idea as in the case when $j = 1$, we get $y_{k+j} = y_{r+j}$ for each $j \geq 1$. We can find the smallest n such that $y_{n+1} = y_{s+1}$ for some $s > n$ (then, $1 \leq n < k$) and choose the smallest $m > n$ such that $y_{n+1} = y_{m+1}$. Thus, the continued fraction is periodic of length m with initial block of length n. □

Theorem 4. *Suppose α is an irrational number. The $\mathscr{F}_{p^l}$-continued fraction of α is periodic if and only if α is a quadratic surd.*

Proof. Suppose the $\mathscr{F}_{p^l}$-continued fraction of α is periodic and given by

$$x = \frac{1}{0+} \frac{p^l}{b+} \frac{\varepsilon_1}{a_1+} \cdots \frac{\varepsilon_n}{a_n+} \frac{\varepsilon_{n+1}}{a_{n+1}+} \cdots \frac{\varepsilon_{n+m}}{a_{n+m}+} \frac{\varepsilon_{n+1}}{a_{n+1}+} \cdots \frac{\varepsilon_{n+m}}{a_{n+m}+} \frac{\varepsilon_{n+1}}{a_{n+1}+} \cdots, \tag{20}$$

where $n \geq 0$ and $m \geq 1$. Then, $y_{n+1} = y_{(n+mk)+1}$, for $k \geq 0$. By Theorem 1 (6), for $i \geq 0$,

$$\alpha = \frac{x_{i+1} P_i + \varepsilon_{+1} P_{i-1}}{x_{i+1} Q_i + \varepsilon_{i+1} Q_{i-1}}, \tag{21}$$

where P_i/Q_i is the i-th convergent, $x_i = 1/|y_i|$, and y_i is the i-th fin of the $\mathscr{F}_{p^l}$-continued fraction of α with maximum $+1$. Therefore, $y_{i+1} = P_i - \alpha Q_i / \alpha Q_{i-1} - P_{i-1}$. Since $y_{n+1} = y_{(n+mk)+1}$, we get

$$\frac{P_n - \alpha Q_n}{\alpha Q_{n-1} - P_{n-1}} = \frac{P_{n+m} - \alpha Q_{n+m}}{\alpha Q_{(n+m)-1} - P_{(n+m)-1}}, \tag{22}$$

which gives that α is a root of a quadratic polynomial

$$Rx^2 + Sx + t, \tag{23}$$

where $R = Q_{n-1} Q_{n+m} - Q_{n+m-1} Q_n$, $S = (Q_n P_{n+m-1} - P_{n+m} Q_{n-1} + P_n Q_{n+m-1} - P_{n-1} Q_{n+m})$, and $T = P_n P_{n+m-1} + P_{n-1} P_{n+m}$. We have assumed that α is irrational, so it is a quadratic surd. For the converse part, let us assume that α is a quadratic surd. Then, by Lemma 2, $y_1 = p^l \alpha - b$ is also a quadratic surd. Thus, there exists $0 \neq R_0 \in \mathbb{Z}$ and $S_0, T_0 \in \mathbb{Z}$ such that

$$R_0 y_1^2 + S_0 y_1 + T_0 = 0. \tag{24}$$

Let the $\mathscr{F}_{p^l}$-continued fraction of α is given by

$$\alpha = \frac{1}{0+} \frac{p^l}{b+} \frac{\varepsilon_1}{a_1+} \frac{\varepsilon_2}{a_2+} \cdots \frac{\varepsilon_n}{a_n+} \cdots. \tag{25}$$

Then, the semi-regular continued fraction

$$y_1 = \frac{\varepsilon_1}{a_1+} \frac{\varepsilon_2}{a_2+} \cdots \frac{\varepsilon_n}{a_n+} \cdots. \tag{26}$$

Let P_k/Q_k and A_k/B_k denote the k-th convergent of the $\mathscr{F}_{p^l}$-continued fraction of α and the corresponding continued fraction of y_1, respectively. Then, $P_k = bB_k + A_k$ and $Q_k = p^l B_k$. If y_k is the fin at the k-th stage for $k \geq 1$, then for $k \geq 1$,

$$y_1 = \frac{A_k + y_{k+1} A_{k-1}}{B_k + y_{k+1} B_{k-1}}. \tag{27}$$

Replacing the value of y_1 in (24), we get

$$
\begin{aligned}
&R_0\left(\frac{A_k + y_{k+1}A_{k-1}}{B_k + y_{k+1}B_{k-1}}\right)^2 + S_0\left(\frac{A_k + y_{k+1}A_{k-1}}{B_k + y_{k+1}B_{k-1}}\right) + T_0 = 0,\\
&R_0\left(A_k + y_{k+1}A_{k-1}\right)^2 + S_0\left(A_k + y_{k+1}A_{k-1}\right)\left(B_k + y_{k+1}B_{k-1}\right) + T_0\left(B_k + y_{k+1}B_{k-1}\right)^2 = 0,\\
&R_k y_{k+1}^2 + S_k y_{k+1} + T_k = 0,
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
R_{k+1} &= R_0A_{k-1}^2 + S_0A_{k-1}B_{k-1} + T_0B_{k-1}^2,\\
S_{k+1} &= 2A_kA_{k-1}R_0 + (A_kB_{k-1} + B_kA_{k-1})S_0 + 2B_kB_{k-1}T_0,\\
T_{k+1} &= R_0A_k^2 + S_0A_kB_k + T_0B_k^2.
\end{aligned}
\tag{29}
$$

For $k \geq 1$,

$$
S_{k+1}^2 - 4R_{k+1}T_{k+1} = S_0^2 - 4R_0T_0. \tag{30}
$$

Thus, the discriminant remains unchanged for each k. We note that $R_{k+1} = T_k$. If for a natural number k, T_k, and T_{k+1} are bounded, then R_k and S_k are also bounded since the discriminant is bounded. Now, we claim that T_k is bounded for every $k \in K_\alpha$, where

$$
K_\alpha = \left\{k \in \mathbb{N} \middle| \frac{\varepsilon_{k+1}}{a_{k+1}} \neq \frac{-1}{2} \text{ in the } \mathcal{F}_{p^l} - \text{continued fractionof } \alpha\right\}. \tag{31}
$$

By Corollary 1, the cardinality of the set K_α is infinite. Let $k^* \in K_\alpha$, then by Lemma 1

$$
\begin{aligned}
&\left|Q_{k^*}\alpha - P_{k^*}\right| < \frac{p^l}{Q_{k^*}}\\
&\left|B_{k^*}y_1 - A_{k^*}\right| < \frac{1}{B_{k^*}}\left(\text{since } y_1 = p^l\alpha - b, P_k^* = bB_k^* + A_k^* \text{ and } Q_k^* = p^lB_k^*\right).
\end{aligned}
\tag{32}
$$

We can write $A_{k^*} = B_{k^*}y_1 + \delta/B_{k^*}$, for some δ with $|\delta| < 1$. Using this value, we get

$$
\left|T_{k^*+1}\right| < \left|2R_0y_1\right| + \left|S_0\right| + \left|R_0\right|. \tag{33}
$$

Hence, T_{k^*+1} is bounded. Now, we claim that T_{k^*} is also bounded for $k^* \in K_\alpha$. If $\varepsilon_{k^*}/a_{k^*} \neq -1/2$, then $k^* - 1 \in K_\alpha$, and we are done. So, let $\varepsilon_{k^*}/a_{k^*} = -1/2$. If $y_{k^*+1} > 0$, then $x_{k^*} > 2$ as $y_{k^*+1} = x_{k^*} - a_{k^*}$ and

$$
\begin{aligned}
\left|B_{k^*-1}y_1 - A_{k^*-1}\right| &= \left|B_{k^*-1}\left(\frac{A_{k^*-1} - y_{k^*}A_{k^*-2}}{B_{k^*-1} + y_{k^*}B_{k^*-2}}\right) - A_{k^*-1}\right|\\
&= \frac{1}{\left|B_{k^*-1} + y_{k^*}B_{k^*-2}\right|} \text{since } A_{k^*-1}B_{k^*-2} - A_{k^*-2}B_{k^*-1} = \pm 1\\
&= \frac{1}{\left|x_{k^*}B_{k^*-1} - B_{k^*-2}\right|} < \frac{1}{B_{k^*-1}}.
\end{aligned}
\tag{34}
$$

Now, suppose $y_{k^*+1} < 0$, then $a_{k^*+1} \geq 3$ so that $x_{k^*+1} > 2$ (reasoning is the same as in Lemma 1, and the fact that y_k is irrational) and equivalently $|y_{k^*+1}| < 1/2$. We know that $1/|y_{k^*}| - 2 = y_{k^*+1}$ and $|y_{k^*+1}| < 1$; therefore, $3/2 < 1/|y_{k^*}| < 5/2$. Using this inequality, we get

$$
\left|B_{k^*-1}y_1 - A_{k^*-1}\right| = \frac{1}{\left|B_{k^*-1}x_{k^*} - B_{k^*-2}\right|} < \frac{2}{B_{k^*-1}}. \tag{35}
$$

We apply the same method to get the boundedness of T_{k^*} as in the case of T_{k^*+1}, for each $k^* \in K_\alpha$. Thus, we get R_{k+1}, S_{k+1}, and T_{k+1} are bounded for infinitely many k, that is, for all $k \in K_\alpha$, and the discriminant remains unchanged. But there are only finitely many polynomials with a given discriminant and bounded coefficients. Thus, the sequence y_{k+1} with $k \in K_\alpha$ has entries from a finite set. Thus, there exist integers $r, s \in \mathbb{N}$ with $r < s$ such that $y_{r+1} = y_{s+1}$. The result is achieved by Lemma 3. □

Theorem 5. *Suppose α is a quadratic surd with $0 < \alpha < 1/p^{l-1}$. Then, $\mathcal{F}_{p^l}$-continued fraction of α is purely periodic if and only if $\overline{\alpha} < 0$.*

Proof. Suppose α is a quadratic surd with $0<\alpha<1/p^{l-1}$ and $\overline{\alpha}<0$. Let us assume that the $\mathscr{F}_{p^l}$-continued fraction of α is not purely periodic and it is given by

$$\alpha=\frac{1}{0+}\frac{p^l}{b+}\frac{\varepsilon_1}{a_1+}\cdots\frac{\varepsilon_m}{a_n+}\frac{\varepsilon_{n+1}}{a_{n+1}+}\cdots\frac{\varepsilon_{n+m}}{a_{n+m}+}\frac{\varepsilon_{n+1}}{a_{n+1}+}\frac{\varepsilon_{n+2}}{a_{n+2}+}\cdots\frac{\varepsilon_{n+m}}{a_{n+m}+}\cdots, \tag{36}$$

where $n\geq 1, m\geq 1$ with $y_n\neq y_{n+m}$, and $y_{n+i}=y_{n+m+i}$ for $i\geq 1$. Thus, for $i\geq 0$,

$$\begin{aligned}\overline{\alpha}&=\frac{1}{0+}\frac{p^l}{b+}\frac{\varepsilon_1}{a_1+}\cdots\frac{\varepsilon_i}{a_i+\overline{y_{i+1}}}=\frac{P_i+\overline{y_{i+1}}P_{i-1}}{Q_i+\overline{y_{i+1}}Q_{i-1}},\\ \overline{y_{i+1}}&=\frac{P_i-Q_i\overline{\alpha}}{Q_{i-1}\overline{\alpha}-P_{i-1}}.\end{aligned} \tag{37}$$

We know that $P_i>0$, (since $\alpha>0$) which gives that $\overline{y_{i+1}}<0$ for $i\geq 0$. Furthermore, we claim that $\overline{y_{i+1}}<-1$. Note that $P_i=a_iP_{i-1}+\varepsilon_iP_{i-2}$, and $P_i\geq b\geq 1, \forall i\geq 0$, so that $P_i\geq P_{i-1}$. Suppose $-1<\overline{y_{i+1}}<0$, then $-1<P_i-Q_i\overline{\alpha}/Q_{i-1}\overline{\alpha}-P_{i-1}<0$, but $P_{i-1}>0$, $Q_i>Q_{i-1}$ and $\overline{\alpha}<0$ give that $P_{i-1}>P_i$, which is not possible. Thus, $\overline{y_{i+1}}<-1$ for $i\geq 0$. Since $\overline{y_{n+1}}=\overline{y_{n+m+1}}$, we get

$$\frac{\varepsilon_n}{\overline{y_n}}-\frac{\varepsilon_{n+m}}{\overline{y_{n+m}}}=a_{n+m}-a_n. \tag{38}$$

We note that $\overline{y_n}<-1$ and $\overline{y_{n+m}}<-1$, and so $-2<\varepsilon_n/\overline{y_n}-\varepsilon_{n+m}/\overline{y_{n+m}}<2$. The RHS. of (38) is an integer. We split the discussion into two cases. First, suppose $a_{n+m}\neq a_n$, then without the loss of generality, we may assume that $\varepsilon_n/y_n-\varepsilon_{n+m}/y_{n+m}=1$. We know that $|y_n|<1$, and hence, we get $\varepsilon_n=1=\varepsilon_{m+n}$. By (38), we get $\overline{y_n}=\overline{y_{n+m}}/\overline{y_{n+m}}+1$, but $\overline{y_n}<-1$ and $\overline{y_{n+m}}/\overline{y_{n+m}}+1>0$, which is not possible. Now, suppose $a_n=a_{n+m}$, then $\varepsilon_n\neq\varepsilon_{n+m}$. Again, by (38),

$$\frac{\varepsilon_n}{\overline{y_n}}=\frac{\varepsilon_{n+m}}{\overline{y_{n+m}}}, \tag{39}$$

which implies that $\overline{y_n}$ and $\overline{y_{n+m}}$ have different signs; hence, we get a contradiction.

Now, for the converse part, we assume that α with $0<\alpha<1/p^l$ has a purely periodic continued fraction. By Theorem 4, we know that α is a quadratic surd. Then, there exists a positive integer m such that $p^l\alpha-b=y_{m+1}$ with

$$\alpha=\frac{P_m+y_{m+1}P_{m-1}}{Q_m+y_{m+1}Q_{m-1}}, \tag{40}$$

and so $p^lQ_{m-1}\alpha^2+(Q_m-bQ_{m-1}-p^lP_{m-1})\alpha+(bP_{m-1}-P_m)=0$. If $(Q_m-bQ_{m-1}-p^lP_{m-1})/p^lQ_{m-1}<0$, then we are done. Let us suppose $(Q_m-bQ_{m-1}-p^lP_{m-1})/p^lQ_{m-1}>0$. Then, $Q_m/p^lQ_{m-1}>b/p^l+P_{m-1}/Q_{m-1}>2b-1/p^l$ and so $a_m\geq 2b-1$ when $\varepsilon_m=1$ and $a_m\geq 2b$, when $\varepsilon_m=-1$. Using values of a_m and ε_m, we get $bP_{m-1}-P_m<0$, and hence $\overline{\alpha}<0$. □

Let D be a positive integer which is not a perfect square; then, the irrational conjugate of $\sqrt{D}$ is negative. Hence, we have the following corollary.

Corollary 3. *Suppose D is a positive integer which is not a perfect square. Then, the $\mathscr{F}_{p^l}$-continued fraction of $\sqrt{D}$ is purely periodic.*

The following proposition record the pattern of partial numerator ε_i and denominator a_i in the $\mathscr{F}_{p^l}$-continued fraction expansion of $\sqrt{D}$.

Proposition 2. *Suppose D is a positive integer which is not a perfect square. Let m be the period length of the $\mathscr{F}_{p^l}$-continued fraction of $\sqrt{D}$. Then, for $m=1$, $a_1=2b$ with $\varepsilon_1=p^{2l}D-b^2$ and for $m>1$, $a_m=2b$, $\varepsilon_{1+i}=\varepsilon_{m-i}$, and $a_i=a_{m-i}$ for an integer i, $1\leq i\leq m/2$.*

Proof. Suppose $m=1$. Then $y_1=p^l\sqrt{D}-b$ so that

$$\sqrt{D}=\frac{1}{0+}\frac{p^l}{b+}\frac{\varepsilon_1}{a_1+\left(p^l\sqrt{D}-b\right)}. \tag{41}$$

Thus $\sqrt{D}$ is a root of the following polynomial:

$$p^{2l}x^2+(a_1-2b)p^lx+\left(b^2-a_1b-\varepsilon_1\right), \tag{42}$$

and hence, $a_1-2b=0$; equivalently, $a_1=2b$. Using the value of a_1, we get $\varepsilon_1=p^{2l}D-b^2$. Now, suppose $m>1$. Then,

$$p^l\sqrt{D}-b=\frac{\varepsilon_1}{a_1+}\frac{\varepsilon_2}{a_2+}\cdots\frac{\varepsilon_m}{a_m+\left(p^l\sqrt{D}-b\right)}. \tag{43}$$

Let y_i denotes the fin at the i-th stage, then

$$\begin{aligned}p^l\sqrt{D}-b&=y_1=\frac{\varepsilon_1}{a_1+y_2}, y_2=\frac{\varepsilon_2}{a_2+y_3},\ldots,y_m\\ &=\frac{\varepsilon_m}{a_m+y_1}.\end{aligned} \tag{44}$$

For $i\geq 1$, the number x_i is given by

$$x_i=\frac{\varepsilon_i}{y_i}=a_i+\frac{\varepsilon_{i+1}}{a_{i+1}+}\frac{\varepsilon_{i+2}}{a_{i+2}}\cdots. \tag{45}$$

Then,

$$x_1=a_1+\frac{\varepsilon_2}{x_2}, x_2=a_2+\frac{\varepsilon_3}{x_3},\ldots,x_m=a_m+\frac{\varepsilon_1}{x_1}, \tag{46}$$

and equivalently,

$$\frac{-\varepsilon_2}{\overline{x_2}}=a_1-\overline{x_1},\frac{-\varepsilon_3}{\overline{x_3}}=a_2-\overline{x_2},\ldots,\frac{-\varepsilon_1}{\overline{x_1}}=a_m-\overline{x_m}. \tag{47}$$

Thus,

$$\frac{-\varepsilon_1}{\overline{x_1}}=a_m+\frac{\varepsilon_m}{a_{m-1}+}\frac{\varepsilon_{m-1}}{a_{m-2}+}\cdots\frac{\varepsilon_2}{a_1-\overline{x_1}}. \tag{48}$$

Note that $-\varepsilon_1/\overline{x_1}=p^l\sqrt{D}+b$, or say, $-\varepsilon_1/\overline{x_1}-2b=p^l\sqrt{D}-b$. Using (43) and (48), we get $a_m=2b, \varepsilon_m=\varepsilon_1$. Furthermore, using the fact that every irrational has a unique $\mathscr{F}_{p^l}$-continued fraction with maximum +1, we get

$$\varepsilon_{1+i}=\varepsilon_{m-i}\text{ and }a_i=a_{m-i}, \tag{49}$$

for an integer i with $1\leq i\leq m/2$. □

4. Pell Equation

In this section, D denotes a positive integer, which is not a perfect square. By Corollary 3, the $\mathcal{F}_{p^l}$-continued fraction is purely periodic. For $i \geq 0$, P_i/Q_i denotes the i-th convergent of the $\mathcal{F}_{p^l}$-continued fraction of $\sqrt{D}$ with maximum $+1$. The following theorem states that certain $\mathcal{F}_{p^l}$-convergence of $\sqrt{D}$ serve as a solution to $X^2 - DY^2 = 1$.

Theorem 6. *Suppose the $\mathcal{F}_{p^l}$-continued fraction of $\sqrt{D}$ is periodic of length m.*

(1) If $m = 1$, then

(a) If $\varepsilon_1 = -1$, each P_i/Q_i is a solution to the Pell equation $X^2 - DY^2 = 1$ for $i \geq 0$

(b) If $\varepsilon_1 = 1$, each P_{2i+1}/Q_{2i+1} is a solution to the Pell equation $X^2 - DY^2 = 1$ for $i \geq 0$

(2)

(a) If $m(>1)$ is an odd number, then P_{2mk-1}/Q_{2mk-1} is a solution to the Pell equation $X^2 - DY^2 = 1$, for every $k \geq 1$

(b) If $m(>1)$ is an even integer, then P_{mk-1}/Q_{mk-1} is a solution to the Pell equation $X^2 - DY^2 = 1$, for every $k \geq 1$

Proof. Suppose the $\mathcal{F}_{p^l}$-continued fraction expansion of $\sqrt{D}$ is given by

$$\sqrt{D} = \frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1+}\cdots\frac{\varepsilon_m}{a_m+}\,\frac{\varepsilon_1}{a_1+}\cdots\frac{\varepsilon_m}{a_m+}\,\frac{\varepsilon_1}{a_1+}\cdots. \tag{50}$$

If $m = 1$, then by Proposition 2, $P_0^2 - DQ_0^2 = -\varepsilon_1$. Furthermore, we can write

$$\sqrt{D} = \frac{1}{0+}\,\frac{p^l}{b+}\,\frac{\varepsilon_1}{a_1 + \left(p^l\sqrt{D} - b\right)} \text{ or } \sqrt{D} = \frac{P_1 + \left(p^l\sqrt{D} - b\right)P_0}{Q_1 + \left(p^l\sqrt{D} - b\right)Q_0}. \tag{51}$$

On comparing rational and irrational parts, we get

$$P_1 = p^{2l}D + b^2, \text{ and } Q_1 = 2bp^l, \tag{52}$$

so that $P_1^2 - DQ_1^2 = (b^2 - p^{2l}D)^2 = \varepsilon_1^2$. Now, suppose the result is true up to some $i > 1$, that is, $P_i^2 - DQ_i^2 = \pm 1$. Again,

$$\sqrt{D} = \frac{P_{i+1} + \left(p^l\sqrt{D} - b\right)P_i}{Q_{i+1} + \left(p^l\sqrt{D} - b\right)Q_i}. \tag{53}$$

On comparing rational and irrational parts, we get $P_{i+1} = bP_i + p^l DQ_i$ and $Q_{i+1} = bQ_i + p^l P_i$ so that

$$P_{i+1}^2 - DQ_{i+1}^2 = \left(P_i^2 - DQ_i^2\right)\left(b^2 - p^{2l}D\right) = -\varepsilon_1\left(P_i^2 - DQ_i^2\right). \tag{54}$$

If $\varepsilon_1 = -1$, using induction hypothesis, we get that $P_i^2 - DQ_i^2 = 1$ for $i \geq 0$. Suppose $\varepsilon_1 = 1$, we note that $P_0^2 - DQ_0^2 = -1$ and $P_1^2 - DQ_1^2 = 1$. By the induction hypothesis, we assume that $P_{2i-1}^2 - DQ_{i-1}^2 = 1$ and $P_{2i}^2 - DQ_{2i}^2 = -1$. Using the relation given in (54), we get $P_{2i+1}^2 - DQ_{2i+1}^2 = 1$ and $P_{2(i+1)}^2 - DQ_{2(i+1)}^2 = -1$, for $i \geq -1$. Now, suppose $m > 1$. Then, for $k \geq 1$,

$$\sqrt{D} = \frac{P_{mk} + \left(p^l\sqrt{D} - b\right)P_{mk-1}}{Q_{mk} + \left(p^l\sqrt{D} - b\right)Q_{mk-1}}. \tag{55}$$

We get $Q_{mk} = bQ_{mk-1} + p^l P_{mk-1}$ and $P_{mk} = p^l DQ_{mk-1} + bP_{mk-1}$ so that

$$\pm p^l = Q_{mk}P_{mk-1} - P_{mk}Q_{mk-1} = p^l\left(P_{mk-1}^2 - DQ_{mk-1}^2\right), \tag{56}$$

and hence, $P_{mk-1}^2 - DQ_{mk-1}^2 = \pm 1$ for each $k \geq 1$. Set $B = p^l(P_{mk-1}^2 - DQ_{mk-1}^2)$. If m is even, say $m = 2m'$, then

$$\begin{aligned} B &= Q_{mk}P_{mk-1} - P_{mk}Q_{mk-1} \\ &= (a_{mk}Q_{mk-1} + \varepsilon_{mk}Q_{mk-2})P_{mk-1} - (a_{mk}P_{mk-1} + \varepsilon_{mk}P_{mk-2})Q_{mk-1} \\ &= \varepsilon_{mk}\left(P_{mk-1}Q_{mk-2} - Q_{mk-1}P_{mk-2}\right) \\ &\vdots \\ &= \varepsilon_m\varepsilon_{m-1}\cdots\varepsilon_{m'+1}\varepsilon_{m'}\cdots\varepsilon_2\varepsilon_1\left(Q_0P_{-1} - P_0Q_{-1}\right) \\ &= \varepsilon_1\varepsilon_2\cdots\varepsilon_{m'}\varepsilon_{m'}\cdots\varepsilon_2\varepsilon_1\left(p^l\right). \end{aligned} \tag{57}$$

Thus, $(P_{mk-1}^2 - DQ_{mk-1}^2) = 1$, if m is even. Now, suppose m is odd and set $B' = p^l(P_{2km-1}^2 - DQ_{2km-1}^2)$. Then,

$$
\begin{aligned}
B' &= Q_{2km}P_{2km-1} - P_{2km}Q_{2km-1} \\
&= (a_{2km}Q_{2km-1} + \varepsilon_{2km}Q_{2km-2})P_{2km-1} - (a_{2km}P_{2mk-1} + \varepsilon_{2mk}P_{2mk-2})Q_{2mk-1} \\
&= \varepsilon_{2mk}\left(P_{2mk-1}Q_{2mk-2} - Q_{2mk-1}P_{2mk-2}\right) \\
&\vdots \\
&= \varepsilon_m\varepsilon_{m-1}\cdots\varepsilon_1\varepsilon_m\cdots\varepsilon_2\varepsilon_1\left(Q_0P_{-1} - P_0Q_{-1}\right) \\
&= \varepsilon_m^2\varepsilon_{m-1}^2\cdots\varepsilon_1^2\left(p^l\right).
\end{aligned} \tag{58}
$$

Thus, $(P_{2mk-1}^2 - DQ_{2mk-1}^2) = 1$ for each $k \geq 1$ when m is odd. □

Lemma 4. *Suppose $0 < K \leq p^l/2$. Let $r/p^l s \in \mathcal{X}_{p^l}$ be such that*

$$\left|p^l s\alpha - r\right| < \frac{K}{p^l s}. \tag{59}$$

Then, $r/p^l s$ is an $\mathcal{F}_{p^l}$-convergent of α.

Proof. Suppose $u/p^l v \in \mathcal{X}_{p^l}$ with $0 < v \leq s$ and $|p^l v\alpha - u| < |p^l s\alpha - r|$. Then,

$$\left|p^l v\alpha - u\right| < \frac{K}{p^l s}. \tag{60}$$

We have

$$\frac{1}{p^l vs} \leq \left|\frac{u}{p^l v} - \frac{r}{p^l s}\right| \leq \left|\alpha - \frac{u}{p^l v}\right| + \left|\alpha - \frac{r}{p^l s}\right| < \frac{K}{p^{2l} sv} + \frac{K}{p^{2l} s^2}. \tag{61}$$

Thus, $q > s(p^l/K - 1)$. By assumption $0 < K < p^l/2$, and so $v > s$, which yields a contradiction. Thus, $u/p^l v \in \mathcal{X}_{p^l}$ with $0 < v \leq s$ and $|p^l v\alpha - u| \geq |p^l s\alpha - r|$ so that $r/p^l s$ is the best approximation of α by an element of $\mathcal{X}_{p^l}$ and hence an $\mathcal{F}_{p^l}$-convergent of α. □

Theorem 7. *Let D be a positive integer which is not a perfect square. Suppose $(X, Y) \in \mathbb{Z} \times \mathbb{Z}$ is a solution of the Pell equation $X^2 - DY^2 = \pm 1$ with $Y \in p^l\mathbb{Z}$. Then, X/Y is a convergent of the $\mathcal{F}_{p^l}$-continued fraction of $\sqrt{D}$ with maximum $+1$.*

Proof. Suppose $(P, p^l Q)$ is a solution to $X^2 - DY^2 = 1$, then

$$
\begin{aligned}
P^2 - p^{2l}DQ^2 &= 1, \\
\left(P - p^l Q\sqrt{D}\right)\left(P + p^l Q\sqrt{D}\right) &= 1, \\
\left(P - p^l Q\sqrt{D}\right)^2 + \left(P - p^l Q\sqrt{D}\right)2p^l Q\sqrt{D} &= 1, \\
\left(P - p^l Q\sqrt{D}\right)p^l Q &< \frac{1}{2\sqrt{D}}.
\end{aligned} \tag{62}
$$

We note that $P - p^l Q\sqrt{D} > 0$, hence by Lemma 4, $P/p^l Q$ is an $\mathcal{F}_{p^l}$-convergent of $\sqrt{D}$ (since $1/2\sqrt{D} < 1$). □

Lemma 5. *Suppose P_i/Q_i denotes the i-th convergent of the $\mathcal{F}_{p^l}$-continued fraction of $\sqrt{D}$ with maximum $+1$. Then,*

(1) $P_i^2 - DQ_i^2 = P_{km+i}^2 - DQ_{mk+i}^2$, *for* $0 \leq i \leq (m-1)$

(2) $|P_i^2 - DQ_i^2| = 1$ *if and only if* $i = mk - 1$, *for some* $k \in \mathbb{N}$

(3) $|P_i^2 - DQ_i^2| = |P_{m-(i+2)}^2 - DQ_{m-(i+2)}^2|$, *for* $0 \leq i \leq \lfloor m/2 \rfloor - 1$

Proof. Suppose $i \geq 0$, the $i+1$-th fin is given by

$$y_{i+1} = \frac{\sqrt{D}Q_i - P_i}{P_{i-1} - \sqrt{D}Q_{i-1}}. \tag{63}$$

We can write y_{i+1} in the following way:

$$y_{i+1} = \frac{M_{i+1} + p^l\sqrt{D}}{N_{i+1}}, \tag{64}$$

where $M_{i+1} = \pm(P_iP_{i-1} - DQ_iQ_{i-1})$ and $N_{i+1} = \pm(P_{i-1}^2 - DQ_{i-1}^2)$. Since the continued fraction of $\sqrt{D}$ is purely periodic of length m, $y_i = y_{km+i}$, $\forall 1 \leq i \leq m$ and $k \geq 0$. On comparing the rational and irrational parts, we get

$$M_i = M_{mk+i} \text{ and } N_i = N_{mk+i}. \tag{65}$$

Thus, $P_{i-1}^2 - DQ_{i-1}^2 = P_{mk+(i-1)}^2 - DQ_{mk+(i-1)}^2$, $\forall 1 \leq i \leq m$ and $k \geq 0$, and we get the first statement. Now, suppose $|P_i^2 - DQ_i^2| = 1$ so that $|N_{i+2}| = 1$. Then,

$$\left|y_{i+2}\right| = \left|M_{i+2} + p^l\sqrt{D}\right| < 1, \tag{66}$$

and hence, $-M_{i+2} - 1 < p^l\sqrt{D} < -M_{i+2} + 1$. For each i, notice that M_i is an integer coprime to p. Thus, the above inequality gives that $M_{i+2} = -b$ so that

$$y_{i+2} = p^l\sqrt{D} - b = y_{mk+1}, \tag{67}$$

for each $k \geq 0$. Thus, we get $i + 2 = mk + 1$, equivalently, $i = mk - 1$. The converse part the second statement is clear from the proof of Theorem 6. For the third statement, recall that

$$y_{m-(i+1)} = \frac{\varepsilon_{i+2}\left(P_i + \sqrt{D}Q_i\right)}{P_{i+1} + \sqrt{D}Q_{i+1}}. \tag{68}$$

Now, we can write

$$P_{m-(i+2)}^2 - DQ_{m-(i+2)}^2 = \left(P_{m-(i+2)} + \frac{\varepsilon_{i+2}\left(P_i + \sqrt{D}Q_i\right)}{P_{i+1} + \sqrt{D}Q_{i+1}} Q_{m-(i+2)} \right) A, \tag{69}$$

where $A = (P_{m-(i+2)} + \sqrt{D}Q_{m-(i+2)})$ and $0 \le i \le \lfloor m/2 \rfloor - 1$. Using the value of $y_{m-(i+1)}$ and comparing the rational and irrational terms, we get

$$B\left(Q_{m-(i+2)}P_{i+1} + Q_{m-(i+3)}P_i\right) = \pm \varepsilon_{i+2}\left(P_i P_{m-(i+2)} + DQ_i Q_{m-(i+2)}\right), \tag{70}$$

$$B\left(Q_{m-(i+2)}Q_{i+1} + Q_{m-(i+3)}Q_i\right) = \pm \varepsilon_{i+2}\left(P_i Q_{m-(i+2)} + DQ_i P_{m-(i+2)}\right), \tag{71}$$

where $B = (P_{m-(i+2)}^2 - DQ_{m-(i+2)}^2)$. By (70) and (71),

$$P_{m-(i+2)}^2 - DQ_{m-(i+2)}^2 = \varepsilon_{i+2}\left(P_i^2 - DQ_i^2\right), \tag{72}$$

and hence $|P_{m-(i+2)}^2 - DQ_{m-(i+2)}^2| = |(P_i^2 - DQ_i^2)|$. □

Combining the results of Theorems 6, 7, and Lemma 5, we obtain our main result which can be stated as follows.

Theorem 8. *Let p be an odd prime and $l \in \mathbb{N}$. Suppose D is a positive integer which is not a perfect square. Then,*

(1) The Pell equation $X^2 - DY^2 = 1$ is always solvable in $\mathbb{Z} \times p^l\mathbb{Z}$

(2) Let P_i/Q_i denote the (i)-th convergent of the $\mathcal{F}_{p^l}$-continued fraction of $\sqrt{D}$ with period m. Then,

(a) If m is even, then the solution set of $X^2 - DY^2 = 1$ is given by

$$\left\{(P_{mk-1}, Q_{mk-1}) | k \in \mathbb{N}\right\}. \tag{73}$$

(b) If $m > 1$ is odd, then the solution set of $X^2 - DY^2 = 1$ is given by

$$\left\{(P_{2mk-1}, Q_{2mk-1}) | k \in \mathbb{N}\right\}. \tag{74}$$

Corollary 4. *The number $1 + Dp^2$ is a complete square if and only if $\mathcal{F}_p$-continued fraction of $\sqrt{D}$ is periodic of length 1.*

Remark 1. In algebraic number theory, Dirichlet's unit theorem states that the group of units with norm 1, say $\mathcal{U}$, of $\mathbb{Z}[\sqrt{D}]$ is an infinite cyclic group. Rewriting the Pell equation as

$$(X + \sqrt{D}Y)(X - \sqrt{D}Y) = 1, \tag{75}$$

it shows that a solution to this equation contributes to a nontrivial unit in $\mathbb{Z}[\sqrt{D}]$. Given a solution (X_1, Y_1), one can find infinitely many (X_n, Y_n) by the following equation:

$$\left(X_n + \sqrt{D}Y_n\right) = \left(X_1 + \sqrt{D}Y_1\right)^n. \tag{76}$$

A solution (X, Y) to Pell equation with the smallest $Y > 0$ serves as a generator of $\mathcal{U}$. Here, we look at a subgroup $\mathcal{U}_{p^l}$ of $\mathcal{U}$ which is given by

$$\mathcal{U}_{p^l} = \left\{X + Y\sqrt{D} \,|\, X + Y\sqrt{D} \in \mathcal{U}, Y \in p^l\mathbb{Z}\right\}. \tag{77}$$

The group $\mathcal{U}_{p^l}$ is a cyclic group of infinite order and the solution (P, Q) to the Pell equation in $\mathbb{Z} \times p^l\mathbb{Z}$ with the smallest $Q > 0$ serves as its generator.

Example 1. The $\mathcal{F}_3$-continued fraction of $\sqrt{5}$ is

$$\frac{1}{0+}\,\frac{3}{7+}\,\frac{-1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{14+}\,\frac{-1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{14+}\cdots. \tag{78}$$

The corresponding set of convergent is

$$\left\{\frac{7}{3}, \frac{20}{9}, \frac{47}{21}, \frac{161}{72}, \frac{2207}{987}, \frac{6460}{2889}, \frac{15127}{6765}, \frac{51841}{23184}, \frac{710647}{317811}, \ldots\right\}. \tag{79}$$

The continued fraction is periodic of length $m = 4$, which is even. The $m-1$-th convergence is 161/72. Then, $(161)^2 - 5(72)^2 = 1$. Thus, we get our first solution to Pell equation in $\mathbb{Z} \times 3\mathbb{Z}$; now, the next solution is given by the 7-th convergence which is 51841/23184. One can check that $(51841, 23184)$ also satisfies the Pell equation. We note that

$$(161 + \sqrt{5}\,72)^2 = 25921 + 23184\sqrt{5} + 25920 = 51841 + 23184\sqrt{5}. \tag{80}$$

Thus, $(51841, 23184)$ is obtained by $(161, 72)$ by comparing the rational and irrational part of $(161 + \sqrt{5}\,72)^2$. Other solutions can be obtained by the rational and irrational part of $(161 + \sqrt{5}\,72)^n$, where $n \in \mathbb{N}$.

Example 2. Let $D = 455$, $p = 3$. Then, $1 + Dp^2 = 4096 = 64^2$. The f_3-continued fraction of $\sqrt{455}$ is

$$\frac{1}{0+}\,\frac{3}{64+}\,\frac{-1}{128+}\,\frac{-1}{128+}\,\frac{-1}{128+}\cdots, \tag{81}$$

which is purely periodic of length 1. If $D = 23$ and $p = 5$, then $1 + Dp^2 = 576 = 24^2$. The $\mathcal{F}_5$ continued fraction of $\sqrt{23}$ is

$$\frac{1}{0+}\,\frac{5}{24+}\,\frac{-1}{48+}\,\frac{-1}{48+}\,\frac{-1}{48+}\cdots, \tag{82}$$

which is again purely periodic of length 1. We know that $46 = 1 + 5 \cdot 3^2$ is not a complete square. The $\mathcal{F}_3$-continued fraction of $\sqrt{5}$ is

$$\frac{1}{0+}\,\frac{3}{7+}\,\frac{-1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{14+}\,\frac{-1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{14+}\cdots, \tag{83}$$

which is purely periodic of length 4 not of length 1.

5. Conclusion

This article gives the complete solution set of the Pell equation $X^2 - DY^2 = 1$ under the condition that Y is a multiple of p^l, where p is a prime and l is a natural number. A solution to the Pell equation with the given restriction can be obtained by the $\mathcal{F}_{p^l}$-continued fraction expansion of $\sqrt{D}$ with maximum +1. Similar to the classical results, this solution set also has a generating element which is nothing but the solution (X, Y) with the smallest $Y > 0$. One direct application to the obtained result is to determine whether for a given prime p and a positive integer D, the number $1 + Dp^2$ is a complete square? The answer is yes if the $\mathcal{F}_p$-continued fraction is periodic of length 1. We believe that the results of this article will be interesting for the readers. One can look for the solutions of the generalized Pell equation with certain restrictions like in [16, 17] with the help of $\mathcal{F}_{p^l}$-continued fractions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declare no conflicts of interest.

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