Indian Institute of Information Technology Allahabad
Linear Algebra (LAL)
C3 Review Test

Program: B.Tech. 1st Semester (IT+ECE)
Duration: **75+15 Minutes** | Full Marks: 50
Date: March 31, 2022 | Time:: 3:00 PM - 4:30 PM

Important Instructions:

1. Attempt all the questions. There is no credit for a solution if the appropriate work is not shown, even if the answer is correct. All the notations are standard and same as used in the lecture notes.
2. Write down your name and enrolment number. Write the solutions clearly with all the steps in details.
3. Submit the solution in PDF format through Google Classroom. **Name the PDF as your LAL_C3_enrolment number**. We will not accept the solution through emails.

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1. Determine whether the following statements are true or false. In either case, give a proper justification (proof or counter example). Do all the parts of this question together, if you wish to do some part later, leave the space for the same. [2*8=16]

   (a) An elementary row operation preserves the rank of a matrix.
   Solution: True:
   Note that a row obtained by applying an elementary row operation is nothing but a linear combination of rows of the matrix, thus an elementary row operation does not change the (row) rank of the matrix. [2]

   (b) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map such that $T(1,0) = (1,4)$ and $T(1,1) = (2,5)$. Then T is one-one.
   Solution: True:
   The set $\{(1,4),(2,5)\} \subset \text{Range}(T)$ is linearly independent. [1]
   Then $\text{Rank}(T) = 2$. Now, by the Rank-Nullity Theorem, we have $\text{Nullity}(T) = 0$.
   That is, T is one-one. [1]

   (c) Let V be a vector space of all $n \times n$ real matrices. Let $T : V \to V$ be a linear map defined by $T(X) = AX - XA$, for a fix $A \in V$. Then T is invertible.
   Solution: False:
   Take $X = I_n$, the identity matrix. Then $T(X) = 0$ but $X = I_n \neq 0$. [1]
   That is, T is not one-one, hence, T is not invertible. [1]

   (d) If $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is similar to $D = \begin{bmatrix} d & 0 \\ 0 & e \end{bmatrix}$ where $a, b, c, d, e \in \mathbb{R}$. Then $b = 0$.

Solution: False:
Take $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. [1]
Since A has distinct eigenvalues, A is diagonalizable. That is, A is similar to a diagonal matrix $D = \begin{bmatrix} d & 0 \\ 0 & e \end{bmatrix}$ for some $d, e \in \mathbb{R}$. [1]

(e) Let $V = \{A \in M_2(\mathbb{R}) : \text{trace}(A) = 0\}$. Then the vector spaces V and $\mathbb{R}^4$ are isomorphic over $\mathbb{R}$.

Solution: False:
The set $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ forms a basis of V. Then $\dim(V) = 3$. [1]
It is well-known that $\dim(\mathbb{R}^4) = 4$. Recall that if two finite-dimensional vector spaces V_1 and V_2 are isomorphic then $\dim(V_1) = \dim(V_2)$. Therefore, V and $\mathbb{R}^4$ cannot be isomorphic. [1]

(f) Let T be a linear operator on an inner product space V such that $\langle T(x), T(x)\rangle = \langle x, x\rangle$ for all $x \in V$. Then T is one-one.

Solution: True:
Let $x, y \in V$. Since T is a linear operator, we have

$$\langle T(x)-T(y), T(x)-T(y)\rangle = \langle T(x-y), T(x-y)\rangle = \langle x-y, x-y\rangle = \|x-y\|^2. \quad [1]$$

Suppose $T(x) = T(y)$. Then, from the above, we obtain $\|x - y\| = 0$. That is, $x = y$. Therefore, T is one-one. [1]

(g) If A is a 3×3 complex matrix such that $A^3 + A = 0$, then $\text{trace}(A) = 0$.
Solution:True:
Recall the properties of eigenvalues mentioned in Lecture-14 that: $\text{trace}(A) = \frac{a_1}{a_0}$, where the characteristic polynomial of A is $f(\lambda) = |\lambda I - A| = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$. [1]
In view of the above, the condition $A^3 + A = 0$ implies that $a_1 = 0$. Therefore, $\text{trace}(A) = 0$. [1]

(h) Let $A \in M_n(\mathbb{R})$. Then AA^T is positive semi-definite.
Solution: True:
The matrix AA^T is symmetric: $(AA^T)^T = (A^T)^T A^T = AA^T$. [1]
Now, the result follows from: $X^T AA^T X = (A^T X)^T (A^T X) = \|A^T X\|^2 \geq 0$. [1]

2. Let $V = P_5(\mathbb{R})$, $W_1 = \{p(x) \in V \mid x^4 p(\frac{1}{x}) = p(x)\}$ and $W_2 = \{p(x) \in V \mid p(-x) = p(x)\}$. Find the dimensions of W_1, W_2 and $W_1 \cap W_2$. [5]

Solution: Let $P_5 = \{p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5\}$.

Then $p(x) = x^4 p(\frac{1}{x}) \Rightarrow a_0 = a_4,\ a_1 = a_3,\ a_5 = 0$. [1]

Therefore $W_1 = \{p(x) = a_0(1 + x^4) + a_1(x + x^3) + a_2x^2\}$.

$\because W_1 = span(S)$, where $S = \{x^2, x + x^3, 1 + x^4\}$ which is LI, $dim(W_1) = 3$. [1]

Similarly $p(-x) = p(x) \Rightarrow a_1 = 0,\ a_3 = 0,\ a_5 = 0.$

Therefore $W_2 = \{p(x) = a_0 + a_2x^2 + a_4x^4\}$. Thus $dim(W_2) = 3$. [1]

$x^4p(\frac{1}{x}) = p(x)$ and $p(-x) = p(x)$ together implies $a_1 = a_3 = a_5 = 0$ and $a_0 = a_4$. [1]

Thus $W_1 \cap W_2 = \{p(x) = a_0 + a_2x^2 + a_0x^4\}$. Thus $dim(W_1 \cap W_2) = 2$. [1]

3. Determine the following for the matrix A. [10]

   (a) Find the eigenvalues.

   (b) Find the bases of corresponding eigenspaces.

   (c) Find the characteristic and minimal polynomials.

   (d) Is the matrix diagonalizable? Justify your answer.

$$A = \begin{bmatrix} 1001 & 3 & 5 & 7 & \cdots & 2n-1 \\ 1 & 1003 & 5 & 7 & \cdots & 2n-1 \\ 1 & 3 & 1005 & 7 & \cdots & 2n-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 3 & 5 & 7 & \cdots & 1000+(2n-1) \end{bmatrix}.$$

Solution: Let $B = A - 1000I$, that is, $A = B + 1000I$. [1]

Then

$$B = \begin{bmatrix} 1 & 3 & 5 & \dots & 2n-1 \\ 1 & 3 & 5 & \dots & 2n-1 \\ 1 & 3 & 5 & \dots & 2n-1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 3 & 5 & \dots & 2n-1 \end{bmatrix}.$$

$\because$ rank$(B) = 1 < n$, B has an eigenvalue $\lambda_{1(B)} = 0$ and using trace, the remaining eigenvalue of B is $\lambda_{2(B)} = n^2$. [1/2+1/2]

Result: if λ is an eigenvalue of a matrix M and $g(x)$ is any polynomial, then $g(\lambda)$ is an eigenvalue of $g(M)$.

Here consider $g(x) = x + 1000$, the eigenvalues of A are:

$\lambda_{1(A)} = \lambda_{1(B)} + 1000 = 1000$ and $\lambda_{2(A)} = \lambda_{2(B)} + 1000 = n^2 + 1000$. [1+1]

Result: if v is an eigenvector of a matrix M corresponding to eigenvalue λ, then v is also eigenvector of $M \pm \mu I$ for μ in the filed.

$\therefore$ The eigenspace corresponding to $\lambda_{1(A)} = 1000$ is the eigenspace corresponding to $\lambda_{1(B)}$, which is given by:

$$\begin{aligned} E_{\lambda_{1(B)}=0}(= E_{\lambda_{1(A)}=1000}) &= \{\bar{X} = (x_1, x_2, \dots, x_n) \,:\, B\bar{X} = \bar{0}\} \\ &= \{\bar{X} \,:\, x_1 + 3x_2 + 5x_3 + \dots + (2n+1)x_n = 0\} \\ &= \{\bar{X} = (-3x_2 - 5x_3 - \dots - (2n+1)x_n, x_2, x_3, \dots, x_n)\} \\ &= \{\bar{X} = x_2v_1 + x_3v_2 + x_4v_3 + \dots + x_nv_{n-1}\}. \end{aligned}$$

Where $S = \{v_1 = (-3, 1, 0, 0, \ldots, 0),\ v_2 = (-5, 0, 1, 0, \ldots, 1),\ v_3 = (-7, 0, 0, 1, \ldots, 0),$ $\ldots,\ v_{n-1} = (-(2n+1), 0, 0, 0, \ldots, 1)\}$. [2]

$\therefore E_{\lambda_{1A}} = \text{span}(S_1)$.

$\because S_1$ is linearly independent, S_1 is a basis of $E_{\lambda_{1(A)}=1000}$.

$E_{\lambda_{2(B)}=n^2} (= E_{\lambda_{2(A)}=1000+n^2}) = \{\bar{X} = (x_1, x_2, \ldots, x_n) : (B - n^2 I)\bar{X} = \bar{0}\}$.

Solving the system $(B - n^2 I)\bar{X} = \bar{0}$, we get:

$$\begin{aligned} (1-n^2)x_1 + 3x_2 + 5x_3 + \cdots + (2n+1)x_n &= 0 \\ x_1 + (3-n^2)x_2 + 5x_3 + \cdots + (2n+1)x_n &= 0 \\ x_1 + 3x_2 + 5x_3 + \cdots + (2n+1)x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_1 + 3x_2 + 5x_3 + \cdots + ((2n+1) - n^2)x_n &= 0 \end{aligned}$$

Then $\bar{X} = k(1, 1, \ldots, 1)$.

Thus $E_{\lambda_{2(A)}=1000+n^2} = \text{span}(S_2)$, where $S_2 = \{(1, 1, \ldots, 1)\}$. [1]

Since S_2 is linearly independent, S_2 is a basis of $E_{\lambda_{2(A)}=1000+n^2}$.

The characteristic polynomial of A is $(x - 1000)^{n-1}(x - (n^2 + 1000))$. [1]

Since AM = GM, the matrix is diagonalizable and [1]

the minimal polynomial of A is therefore $(x - 1000)(x - (n^2 + 1000))$. [1]

4. Let V be an inner product space over $\mathbb{R}$. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V such that, whenever $v = \sum a_i v_i$, then $\|v\|^2 = \sum a_i^2$. Show that $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of V. [4]

   Solution: By hypothesis, $\|v_i\|^2 = 1$. [1]

   For $i \neq j$, $\|v_i + v_j\|^2 = 2$. [1]

   $\Rightarrow \langle v_i + v_j, v_i + v_j \rangle = 2$

   $\Rightarrow \langle v_i, v_i \rangle + \langle v_i, v_j \rangle + \langle v_j, v_i \rangle + \langle v_j, v_j \rangle = 2$ [1]

   $\Rightarrow \|v_i\|^2 + 2\langle v_i, v_j \rangle + \|v_j\|^2 = 2$

   $\Rightarrow \langle v_i, v_j \rangle = 0$. [1]

   Hence $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of V

5. Let $A \in M_{m\times n}(\mathbb{R})$ with rank r. Suppose $A = U\Sigma V^T$ is an SVD of A. Define $A^\dagger = V\Sigma^\dagger U^T$, where $\Sigma^\dagger \in M_{n\times m}(\mathbb{R})$ is given by

$$\Sigma^\dagger_{ij} = \begin{cases} 1/\Sigma_{ii} & i = j, \text{ and } \Sigma_{ii} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

   Then show that

   (a) $\|Uy\| = \|y\|$ for all $y \in \mathbb{R}^m$. [2]

   (b) $\|Ax - b\|^2 = \|\Sigma V^T x - U^T b\|^2$ for $b \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$. [2]

(c) $\|\Sigma z - y\|^2 = \sum_{i=1}^{r}(\Sigma_{ii}z_i - y_i)^2 + \sum_{i=r+1}^{m} y_i^2$ for $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$. [2]

(d) $\|(\Sigma\Sigma^\dagger - I_m)y\|^2 = \sum_{i=r+1}^{m} y_i^2$.for $y \in \mathbb{R}^m$. [4]

(e) $\|AA^\dagger b - b\|^2 = \sum_{i=r+1}^{m}(U^Tb)_i^2$. [3]

(f) $A^\dagger b$ is a least square solution of an inconsistent system $AX = b$. [2]

Solution: (a) Note that $\langle x, x\rangle = x^t x$ for all $x \in \mathbb{R}^n$ and U is orthogonal so that $U^TU = I_n$. [1]

$\|Uy\|^2 = \langle Uy, Uy\rangle = (Uy)^T(Uy) = y^tU^TUy = y^Ty = \|y\|^2$. [1]

(b)

$$\begin{aligned}\|Ax - b\|^2 &= \|U\Sigma V^Tx - b\|^2 \\ &= \|U^T(U\Sigma V^Tx - b)\|^2 \quad \text{using part(a)} && [1] \\ &= \|\Sigma V^Tx - U^Tb\|^2 \quad (\text{ since } U^TU = I_m) && [1]\end{aligned}$$

(c) For any $x \in \mathbb{R}^m$, $\|x\|^2 = \sum_{i=1}^{m} x_i^2$ and for any matrix $C \in M_m(\mathbb{R})$ and $y \in \mathbb{R}^m$, $(Cy)_i = \sum_{j=1}^{m} C_{ij}y_j$ for $1 \leq i \leq m$. [1]

$\|\Sigma z - y\|^2 = \sum_{i=1}^{m}(\sum_{j=1}^{n} \Sigma_{ij}z_j - y_i)^2 = \sum_{i=1}^{r}(\Sigma_{ii}z_i - y_i)^2 + \sum_{i=r+1}^{m} y_i^2$. (since Σ is a rectangular diagonal matrix of rank r) [1]

(d)Put $z = \Sigma^\dagger y$ in part (c). [1]

$\|(\Sigma\Sigma^\dagger - I_m)y\|^2 = \sum_{i=1}^{r}(\Sigma_{ii}(\Sigma^\dagger y)_i - y_i)^2 + \sum_{i=r+1}^{m} y_i^2 = \sum_{i=1}^{r}(\Sigma_{ii}\Sigma_{ii}^\dagger - 1)^2y_i^2 + \sum_{i=r+1}^{m} y_i^2$. (since Σ and $\Sigma^\dagger$ are rectangular diagonal matrices) [1]

$\Sigma_{ii}\Sigma_{ii}^\dagger = 1$ for $i \leq r,$ and 0 otherwise . [1]

$\|(\Sigma\Sigma^\dagger - I_m)y\|^2 = \sum_{i=r+1}^{m} y_i^2$ [1]

(e)

$$\begin{aligned}\|AA^\dagger b - b\|^2 &= \|(U\Sigma V^T)(V\Sigma^\dagger U^T)b - b\|^2 \\ &= \|(U\Sigma\Sigma^\dagger U^T)b - b\|^2 \quad (\text{ since } V^TV = I_n) && [1] \\ &= \|(\Sigma\Sigma^\dagger - I_m)U^Tb\|^2 \quad \text{using part(a)} && [1] \\ &= \sum_{i=r+1}^{m}(U^Tb)_i^2 \quad (\text{ using part (d)} && [1]\end{aligned}$$

(f) $A^\dagger b$ is a least square if and only if $\|AA^\dagger b - b\|^2 \leq \|Ax - b\|^2$ for all $x \in \mathbb{R}^m$. By part (b) and (c), $\|Ax - b\|^2 = c + \sum_{i=r+1}^{m}(U^Tb)_i^2$, where $c \geq 0$. [1]

Now by (e), $\|AA^\dagger b - b\|^2 \leq \|Ax - b\|^2$. [1]