Lecture 8

Basis & Dimension

Definition 1. Let V be a vector space over a field $\mathbb{F}$. A subset S of V is said to be a **basis** of V if the following conditions are satisfied.

1. S is linearly independent set.

2. The linear span $L(S)$ is the vector space V, that is, $L(S) = V$.

Example 2. *1. Let* $V = \mathbb{F}^n(\mathbb{F})$ *and* $B = \{e_1, e_2, \ldots, e_n\}$*, where* $e_i = (0, 0, \ldots, 1, \ldots, 0)$*, (1 at the i-th component and 0 otherwise). The set B is a basis of* $\mathbb{F}^n$ *and is called* **the standard basis of** $\mathbb{F}^n(\mathbb{F})$*.*

2. The set $B = \{e_{ij} \in M_{m\times n} \mid i,j\text{-th entry of } e_{ij} \text{ is 1 and 0 otherwise}\}$ *is the standard basis of* $M_{m\times n}(\mathbb{F})$ *over* $\mathbb{F}$*.*

3. The set $B = \{1, x, x^2, x^3, \ldots, x^n\}$ *is the standard basis of* $P_n(\mathbb{R})$ *over* $\mathbb{R}$*.*

4. The set $B = \{1, x, x^2, x^3 \ldots\}$ *is the standard basis of* $\mathbb{R}[x]$ *over* $\mathbb{R}$*.*

Remark 3. *1. Every vector space has a basis.*

2. The basis of the zero space is the empty set $\emptyset$*.*

3. A basis of a vector space need not be unique, for instance, $\{(1, 1), (1, -1)\}$ *is also a basis of* $\mathbb{R}^2(\mathbb{R})$*.*

4. A vector space V is called a **finite dimensional vector space** *if it has a finite basis, otherwise it is called an* **infinite dimensional space***.*

Theorem 4. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of a vector space V over $\mathbb{F}$. If $B' = \{w_1, w_2, \ldots, w_m\}$, where $(m > n)$. Then B' is a linearly dependent set.

Proof: We will show that there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{F}$, not all of which are 0, such that $\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_m w_m = 0$. Since B is a basis of V, we can write an element of B' as linear combination of elements of B over $\mathbb{F}$ so that

$$w_1 = a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n$$

$$w_2 = a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n$$

$$\vdots$$

$$w_m = a_{m1}v_1 + a_{m2}v_2 + \ldots + a_{mn}v_n$$

Therefore, $\alpha_1(a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n) + \alpha_2(a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n) + \cdots + \alpha_m(a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n) = 0$. Equivalently,

$(\alpha_1 a_{11}+\alpha_2 a_{21}+\ldots+\alpha_m a_{m1})v_1+(\alpha_1 a_{12}+\alpha_2 a_{22}+\ldots+\alpha_m a_{m2})v_2+\ldots+(\alpha_1 a_{1n}+\alpha_2 a_{2n}+\ldots+\alpha_m a_{mn})v_n = 0$. Since $\{v_1, v_2, \ldots, v_n\}$ is a basis of V, we get

$$\begin{aligned} \alpha_1 a_{11} + \alpha_2 a_{21} + \ldots + \alpha_m a_{m1} &= 0 \\ \alpha_1 a_{12} + \alpha_2 a_{22} + \ldots + \alpha_m a_{m2} &= 0 \\ &\vdots \\ \alpha_1 a_{1n} + \alpha_2 a_{2n} + \ldots + \alpha_m a_{mn} &= 0 \end{aligned}$$

Above is a homogeneous system of n equations in m unknowns with $m > n$, therefore the system has a non-zero solution. Thus, $\alpha_i \neq 0$ for some i so that B' is linearly dependent. □

Corollary 5. *Let $V(\mathbb{F})$ be a finite dimensional vector space. The any two bases of V have the same number of elements.*

Definition 6. Let $V(\mathbb{F})$ be a finite dimensional vector space. Then the number of elements in a basis of V is called **dimension** of V and it is denoted as $\dim(V)$.

Example 7. *1.* $\dim(\mathbb{F}^n(\mathbb{F})) = n$; *2.* $\dim(\mathbb{C}(\mathbb{R})) = 2$; *3.* $\dim(M_{m\times n}(\mathbb{F})) = mn$; *4.* $\dim P_n(\mathbb{R}) = n + 1$; *5.* $\mathbb{R}[x]$ *is an infinite dimensional space.*

Remark 8. *Let V be a finite-dimensional vector space and let $n = \dim V$. Then*

1. any subset of V which contains more than n vectors is linearly dependent;

2. no subset of V which contains fewer than n vectors can span V.

Theorem 9. Let $\{v_1, \ldots, v_n\}$ be a basis for a vector space V. Then each vector in V can be expressed uniquely as a linear combination of the basis vectors.

Proof: Let $v \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{F}$ such that $v = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i$. Then $\sum_{i=1}^{n} (\alpha_i - \beta_i)v_i = 0$, but v_i' s are linearly independent so that $\alpha_i - \beta_i = 0$ for each i. Therefore, each vector in V can be expressed uniquely as a linear combination of the basis vectors.

Theorem 10. Let $S = \{v_1, \ldots, v_n\}$ be a linearly independent subset of a vector space V. If $v \notin L(S)$, then $S \cup \{v\}$ is linearly independent.

Proof: Consider the set $S' = S \cup \{v\}$. Let $\alpha v + \alpha_1 v_1 + \ldots + \alpha_n v_n = 0$. It is enough to show that $\alpha = 0$. If $\alpha \neq 0$, then $v \in L(S)$, which is not true. Hence, $\alpha = 0$ so that S' is L.I. $\square$

Theorem 11. Let V be an n dimensional vector space. Then

1. a linearly independent set of n vectors of V is a basis of V;
2. a set of n vectors of V which spans V is a basis of V.

Proof: Let $S = \{v_1, v_2, \ldots, v_n\} \subset V$ be a linearly independent set. It is enough to show that $L(S) = V$. Suppose it is not true and $v \in V \setminus S$. Then the set $S \cup \{v\}$ is L.I. which contradicts the fact that $\dim V = n$. Thus $v \in L(S)$. Therefore S is a basis of V.

Let $S = \{v_1, v_2, \ldots, v_n\} \subset V$ and $L(S) = V$. Suppose S is linearly dependent. Then there exist i such that v_i a is linear combination of rest of the vectors in S. Therefore, the set $S \setminus \{v_i\}$ spans V having $n - 1$ vectors so that $\dim V \leq n - 1$ which contradicts the fact that $\dim V = n$. $\square$

Example 12. *Find a basis and dimension of the solution space of the homogeneous system $Ax = 0$, where* $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & -1 \\ 3 & 11 & 5 \end{pmatrix}$.

Solution: *The RRE form of A is* $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. *The solution set is $\{(2z, -z, z) \mid z \in \mathbb{R}\}$. Any solution is a linear combination of $(2, -1, 1)$ and a singleton set with a non-zero element is linearly independent. Thus $\{(2, -1, 1)\}$ is a basis of the solution space of $Ax = 0$. Hence, the dimension of the solution space is 1.*