Lecture 7

Linear Combination, Linear Span, Linear Dependence & Independence

Definition 1. *Let V be a vector space over a field $\mathbb{F}$. A vector $v \in V$ is said to be a* **linear combination of the vectors** $v_1, v_2, \ldots, v_k \in V$ *if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{F}$ such that*

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k.$$

Example 2. *1. Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$. Let $v_1 = (1,0), v_2(0,1) \in \mathbb{R}^2$. and $(x,y) \in \mathbb{R}^2$. Then every vector (x,y) in $\mathbb{R}^2$ is a linear combination of v_1 and v_2 as $(x,y) = x(1,0) + y(0,1)$.*

2. Let $\mathbb{R}^3(\mathbb{R})$ and $(1,1,1)$, $(1,1,-1) \in \mathbb{R}^3$. Then $(1,1,2)$ is a linear combination of $(1,1,1)$ and $(1,1,-1)$ as $(1,1,2) = \frac{-1}{2}(1,1,1) + \frac{3}{2}(1,1,-1)$. But $(1,-1,0)$ is not a linear combination of $(1,1,1)$ and $(1,1,-1)$. (Verify yourself!)

Definition 3. Let V be a vector space over the field $\mathbb{F}$ and $S \subseteq V$. Then a vector $v \in V$ is said to be a **linear combination of vectors in** S if there exist a positive integer k and scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ in $\mathbb{F}$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k$, where $v_i \in S$.

Example 4. *Consider the vector space $P(\mathbb{R})$ over $\mathbb{R}$. Let $S = \{1, x, x^2, x^3, \ldots\}$. Then every polynomial in $P(\mathbb{R})$ is a linear combination of vectors in S.*

Definition 5. Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$. Then **linear span of** S, denoted as $L(S)$ or $[S]$, is a subset of V defined as $L(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k \,|\, v_i \in S, \alpha_i \in \mathbb{F}\}$. For instance, $L(\{(1,0),(0,1)\}) = \mathbb{R}^2$ and $L(\{(1,1,1),(1,1,-1)\}) = \{((a,a,b)) \mid a,b \in \mathbb{R}\}$.

Theorem 6. *Let S be a non empty subset of a vector space V over $\mathbb{F}$. Then $L(S)$ is the smallest subspace containing S.*

Proof: Let $v \in S$. Then $1.v \in L(S)$ so that S is contained in $L(S)$. Next, we show that $L(S)$ is a subspace of V. Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k$ and $v' = \beta_1 v_1' + \beta_2 v_2' + \ldots + \beta_l v_l'$ belong to $L(S)$. Then for any scalars γ, δ, $\gamma v + \delta v' = \gamma\alpha_1 v_1 + \gamma\alpha_2 v_2 + \ldots + \gamma\alpha_k v_k + \delta\beta_1 v_1' + \delta\beta_2 v_2' + \ldots + \delta\beta_l v_l' \in L(S)$. Thus $L(S)$ is a subspace of V.

Now to show that $L(S)$ is the smallest subspace containing S, it is enough to show that $L(S)$ is a subset of any subspace containing S. Let T be a subspace of V which contains S and $v \in L(S)$. Then $v = \sum_{i=1}^{k} \alpha_i v_i$ for $\alpha_i \in \mathbb{F}$ and $v_i \in S$. Note that $v_i \in S$ implies $v_i \in T$, and hence $v \in T$ as T is a subspace. □

Definition 7. *Let S be a set of vectors in a vector space V over $\mathbb{F}$. The* **subspace spanned by** S, *denoted as $\langle S \rangle$, is defined to be the intersection of all subspaces of V which contain S.*

Theorem 8. $L(S) = \langle S \rangle$.

Definition 9. **The sum** $S_1 + S_2$ **of two subsets** S_1, S_2 *of a vector space* V *over* $\mathbb{F}$ *is given by*

$$S_1 + S_2 = \{v_1 + v_2 \mid v_1 \in S_1, v_2 \in S_2\}.$$

Theorem 10. Let V be a vector space over $\mathbb{F}$ and U and W be two subspaces of V. Then

1. $U + W$ is a subspace of V;
2. $U + W = L(U \cup W)$.

Proof: Let $v, v' \in U + W$. The $v = u + w$ and$v' = u' + w'$ for some $u, u' \in U$ and $w, w' \in W$. Let $\alpha, \beta \in \mathbb{F}$. Then $\alpha v + \beta v' = (\alpha u + \beta u') + (\alpha w + \beta w') \in U + W$. Therefore, $U + W$ is a subspace of V.

Note that $U + W$ is a subspace of V containing $U \cup W$. Hence, $L(U \cup W) \subseteq U + W$. Now suppose $v \in U + W$. Then $v = u + w$, where $u \in U$ and $w \in W$. Note that $u, w \in U \cup W$ and hence, $u + w \in L(U \cup W)$. Therefore, $U + W \subseteq L(U \cup W)$.

Definition 11. Let V be a vector space over $\mathbb{F}$. A subset S of V is said to be **linearly dependent** (LD) if there exist distinct vectors $v_1, v_2, \ldots, v_n \in S$, and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$, not all of which are 0, such that $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$.
A set which is not linearly dependent is called **linearly independent**.

Let $S = \{v_1, v_2, \ldots, v_k\}$. Then $v_1, v_2, \ldots, v_k$ are said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{F}$, not all of which are 0, such that $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$.

The vectors $v_1, v_2, \ldots, v_k$ are not linearly dependent, that is, linearly independent if $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$ implies $\alpha_i = 0$ for all $i = 1, 2, \ldots, k$.

Example 12. *1. Consider the vector space* R^3 *over* $\mathbb{R}$*. The set* $S = \{(n, n, n) \mid n \in \mathbb{N}\}$ *is linearly dependent since* $(2, 2, 2), (3, 3, 3) \in$ *and* $3(2, 2, 2) - 2(3, 3, 3) = 0$ *so that* S *is linearly dependent.*

2. The $S = \{(1, 2, 3), (2, 3, 4), (1, 1, 2)\}$ *is linearly independent in* $\mathbb{R}^3(\mathbb{R})$*. To see this consider* $\alpha_1(1, 2, 3) + \alpha_2(2, 3, 4) + \alpha_3(1, 1, 2) = 0$*. Then* $(\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + 3\alpha_2 + \alpha_3, 3\alpha_1 + 4\alpha_2 + 2\alpha_3) = (0, 0, 0)$*. Thus,* $\alpha_1 + 2\alpha_2 + \alpha_3 = 0, 2\alpha_1 + 3\alpha_2 + \alpha_3 = 0, 3\alpha_1 + 4\alpha_2 + 2\alpha_3 = 0$*. By solving this system of linear equations, we see that* $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$ *is the only possible solution.*

3. Observe that $1.0 = 0$*. Thus, any subset of a vector space containing the zero vector is linearly dependent.*

4. The set $\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\} \subseteq \mathbb{R}^n$ *is linearly independent.*

5. Let $V = \{f \mid f : [-1, 1] \to \mathbb{R}\}$*. The set* $\{x, |x|\}$ *is linearly independent. To see this, consider the equation* $\alpha x + \beta|x| = 0$*. A function is zero if it is zero at every point of the domain.Thus,* $\alpha x + \beta|x| = 0$

for all $x \in [-1, 1]$*. If* $x = 1$ *we get* $\alpha + \beta = 0$ *and if* $x = -1$*,* $\alpha - \beta = 0$*. Solving these two equations we get* $\alpha = \beta = 0$*. Thus the set is linearly independent.*

Remark 13. Let V be a vector space over $\mathbb{F}$. Then

1. the set $\{v\}$ is L.D. if and only if $v = 0$;
2. a subset of a linearly independent set is also linearly independent;
3. a set containing a linearly dependent set is also linearly dependent.