Lecture 5
Determinant Function & Its Properties

Definition 1. *Let $A = (a_{ij})$ be an $n \times n$ matrix and S_n denote the set of all permutation on $S = \{1, 2, \ldots, n\}$. Then determinant is a function from $M_n(\mathbb{F})$ to $\mathbb{F}$, denoted by* $\det(A)$ *or* $|A|$, *and given by*

$$\det(A) = |A| = \sum_{\sigma \in S_n} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

Let $n = 2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then $S_n = \{(1), (1\,2))\}$. Set $\sigma_1 = (1)$ and $\sigma_2 = (1\,2)$. Then

$$\begin{aligned} \det(A) &= sign(\sigma_1) a_{1\sigma_1(1)} a_{2\sigma_1(2)} + sign(\sigma_2) a_{1\sigma_2(1)} a_{2\sigma_2(2)} \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

Properties of Determinant

P1: Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices. Then if B is obtained from A by interchanging two rows of A, then $|A| = -|B|$.

Proof: Let B is obtained by interchanging k-th row and r-th row of A. Then $B = (b_{ij})$ such that $b_{kj} = a_{rj}$, $b_{rj} = a_{kj}$, and $b_{ij} = a_{ij}$ for $j = 1, 2, \ldots, n$ and $i \neq k, r$.

Let $\tau = (k, r)$. Then $S_n = \{\sigma \circ \tau : \sigma \in S_n\}$. Therefore,

$$\begin{aligned} |B| &= \sum_{\sigma \circ \tau} sign(\sigma \circ \tau) b_{1\sigma\circ\tau(1)} \cdots b_{k\sigma\circ\tau(k)} \cdots b_{r\sigma\circ\tau(r)} \cdots b_{n\sigma\circ\tau(n)} \\ &= \sum_{\sigma} sign(\sigma) sign(\tau) b_{1\sigma(1)} \cdots b_{k\sigma(r)} \cdots b_{r\sigma(k)} \cdots b_{n\sigma(n)} \\ &= -\sum_{\sigma} sign(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(r)} \cdots a_{k\sigma(k)} \cdots a_{n\sigma(n)} \qquad \text{(since } sign(\tau) = -1) \\ &= -|A| \end{aligned}$$

P2: If two rows of A are identical, then $|A| = 0$.

Proof: Let $R_1, R_2, \ldots, R_n$ denote the rows of A. It is given that $R_k = R_j$ for some $j \neq k$. Let B be the matrix obtained by interchanging j-th row and k-th row of A. Then $|B| = -|A|$, but $A = B$. Therefore, $|A| = 0$.

P3: If B is obtained by multiplying a row of A by a constant c, then $|B| = c|A|$.

Proof: Let $B = (b_{ij})$ is obtained by multiplying a constant c to the k-th row of A. Then $b_{kj} = ca_{kj}$ and $b_{ij} = a_{ij}$ for $i \neq k$. Then

$$\begin{aligned}
|B| &= \sum_{\sigma} sign(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{k\sigma(k)} \cdots b_{n\sigma(n)} \\
&= \sum_{\sigma} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots ca_{k\sigma(k)} \cdots a_{n\sigma(n)} \\
&= c \sum_{\sigma} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)} \cdots a_{n\sigma(n)} \\
&= c|A|
\end{aligned}$$

P4: Let A, B and C be $n \times n$ matrices which differ only in the k-th row, and $c_{kj} = a_{kj} + b_{kj}$ $\forall j$, then $|C| = |A| + |B|$.

Proof:

$$\begin{aligned}
|C| &= \sum_{\sigma} sign(\sigma) c_{1\sigma(1)} c_{2\sigma(2)} \cdots c_{k\sigma(k)} \cdots c_{n\sigma(n)} \\
&= \sum_{\sigma} sign(\sigma) c_{1\sigma(1)} c_{2\sigma(2)} \cdots (a_{k\sigma(k)} + b_{k\sigma(k)}) \cdots c_{n\sigma(n)} \\
&= \sum_{\sigma} sign(\sigma) c_{1\sigma(1)} c_{2\sigma(2)} \cdots a_{k\sigma(k)} \cdots c_{n\sigma(n)} + \sum_{\sigma} sign(\sigma) c_{1\sigma(1)} c_{2\sigma(2)} \cdots b_{k\sigma(k)} \cdots c_{n\sigma(n)} \\
&= \sum_{\sigma} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)} \cdots a_{n\sigma(n)} + \sum_{\sigma} sign(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{k\sigma(k)} \cdots b_{n\sigma(n)} \\
&= |A| + |B|
\end{aligned}$$

P5: If B is obtained by adding λ times the r-th row of A to its k-th row, then $|A| = |B|$.

Proof: Here, $b_{kj} = \lambda a_{rj} + a_{kj}$, $b_{ij} = a_{ij}$ for $i \neq k$ and $j = 1, 2, \ldots, n$. Then

$$\begin{aligned}
|B| &= \sum_{\sigma} sign(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{k\sigma(k)} \cdots b_{n\sigma(n)} \\
&= \sum_{\sigma} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots (\lambda a_{r\sigma(k)} + a_{k\sigma(k)}) \cdots a_{n\sigma(n)} \\
&= \lambda \sum_{\sigma} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{r\sigma(k)} \cdots a_{n\sigma(n)} + \sum_{\sigma} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots + a_{k\sigma(k)} \cdots a_{n\sigma(n)} \\
&= 0 + |A| = |A|
\end{aligned}$$

P6: Let E be an elementary matrix. Then $|E| \neq 0$.

P7: If E is an elementary matrix, then $|EA| = |E||A|$. (Prove it yourself!)

P8: A is invertible $\Leftrightarrow |A| \neq 0$. (Prove it yourself!)

P9: Let A, B be $n \times n$ matrices. Then $|AB| = |A||B|$.

Proof: Suppose A is not invertible. Then $|A| = 0$. Let $|AB| \neq 0$ so that AB is invertible. Therefore, the system $ABx = 0$ has only trivial solution. But $Ax = 0$ has a non trivial solution, say y. If B is invertible, then $Bx = y$ has a unique solution, say x^*. Note that $x^* \neq 0$ and $ABx^* = 0$ so that $ABx = 0$ has a non-trivial solution which contradicts our assumption and hence, $|AB| = 0$. Now if $|B| = 0$, the system $Bx = 0$ has a non-trivial solution so that $ABx = 0$ has a non-trivial solution which again gives a contradiction. Therefore, $|AB| = 0$.

Suppose A is invertible. Then $A = E_1 \dots E_s$. This implies

$$\begin{aligned} |AB| &= |(E_1 \dots E_s B)| \\ &= |E_1||E_2| \dots |E_s||B| \\ &= |E_1 \dots E_s||B| \\ &= |A||B|. \end{aligned}$$

P10: $|A| = |A^t|$, where A^t denotes the transpose of A.

Remark 2. *The properties P1-P5 are also valid for column operations.*

Cramer's Rule for solving system of linear equations

Let $Ax = b$ be a system of n linear equations in n unknowns such that $|A| \neq 0$. Then the system $Ax = b$ has a unique solution given by

$$x_j = \frac{|C_j|}{|A|},\ j = 1, 2, \dots, n$$

where C_j is the matrix obtained from A by replacing the j-th column of A with the column matrix $b = (b_1, b_2, \dots, b_n)^t$.

Proof: If $|A| \neq 0$, then A is invertible and $x = A^{-1}b$ is the unique solution of $Ax = b$. Define a matrix

$$X_j = \begin{pmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 \\ 0 & 1 & \cdots & x_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & x_n & \cdots & 1 \end{pmatrix}.$$

Note that the matrix $|X_j| = x_j$ (apply properties of determinant function). Therefore,

$$x_j = |X_j| = |I_n X_j| = |A^{-1} A X_j| = \frac{|AX_j|}{|A|} = \frac{|C_j|}{A_j} \, \forall j = 1, 2, \ldots, n.$$