Lecture 3
Elementary Matrices & Row Reduced Echelon Form

Definition 1. Elementary row/column operations: *Let A be an $m \times n$ matrix and $R_1, \ldots, R_m$ denote the rows of A and $C_1, \ldots, C_n$ denote the columns of A. Then an elementary row(column) operation is a map from $M_{m\times n}(\mathbb{F})$ to itself which is any one of the following three types:*

1. *Multiplying the i-th row(column) by a nonzero scalar $\lambda \in \mathbb{F} \setminus \{0\}$ denoted by $R_i \to \lambda R_i$ ($C_i \to \lambda C_i$).*
2. *Interchanging the i-th row(column) and the j-th row(column) denoted by $R_i \leftrightarrow R_j$ ($C_i \leftrightarrow C_j$).*
3. *For $i \neq j$, replacing the i-th row(column) by the sum of the i-th row(column) and μ multiple of the j-th row(column) denoted by $R_i \to R_i + \mu R_j$ ($C_i \to C_i + \mu C_j$).*

A **row operation** is a map from $M_{m\times n}(\mathbb{F})$ to itself which is a composition of finitely many elementary row operations.

Remark 2. 1. *Every elementary row operation is invertible.*
1. The inverse of $R_i \to \lambda R_i$ is $R_i \to \frac{1}{\lambda} R_i$;
2. The inverse of $R_i \leftrightarrow R_j$ is $R_i \leftrightarrow R_j$ (self inverse, i.e., inverse of itself);
3. The inverse of $R_i \to R_i + \mu R_j$ is $R_i \to R_i - \mu R_j$.

2. *Let ρ be an elementary row operation and $A \in M_{m\times n}(\mathbb{F})$. Then $\rho(A) = \rho(I_m)\, A$, where I_m is the $m \times m$ identity matrix.*

Definition 3. Elementary Matrix: *Let I_m denote the $m \times m$ identity matrix. A matrix obtained by performing an elementary row operation on I_m is called an elementary matrix. Therefore, there are three types of elementary matrices:*
1. $E_i(\lambda)$, obtained by multiplying the i-th row by a nonzero scalar $\lambda \in \mathbb{F} \setminus \{0\}$ of I_m.
2. E_{ij}, obtained by interchanging the i-th row and the j-th row of I_m.
3. $E_{ij}(\mu)$, obtained by replacing the i-th row by the sum of the i-th row and μ multiple of the j-th row of I_m.

Example 4.

$$M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

In the above, M_1, M_2 are elementary matrices but M_3 is not.

Remark 5. *1. Performing an elementary row operation on a matrix A is same as pre multiplication of the respective elementary matrix to A.*
2. Performing an elementary column operation on a matrix A is same as post multiplication of the respective elementary matrix to A.

Definition 6. Row-equivalent matrices: *Let A and B be two $m \times n$ matrices over a field $\mathbb{F}$. Then B is said to be row-equivalent to A if B is obtained from A by performing a finite sequence of elementary row operations.*

Theorem 7. If A and B are row equivalent matrices, the homogeneous systems of linear equations $Ax = 0$ and $Bx = 0$ have exactly the same solutions.

Proof: It is given that A and B are row equivalent, that is, there exist elementary row operations, $\rho_1, \rho_2, \ldots, \rho_k$, such that $B = \rho_k \circ \cdots \circ \rho_2 \circ \rho_1(A)$, equivalently,

$$A = A_0 \to A_1 \to \cdots \to A_k = B.$$

It is enough to show that $A_j x = 0$ and $A_{j+1}x = 0$ have the same solutions. In words, elementary operations do not make any change to the solution set. Let $k = 1$. Then $B = \rho(A) \Rightarrow B = \rho(I_m)A$. If $Ax = 0$, then $Bx = \rho(I_m)Ax = \rho(I_m)0 = 0$. Similarly, if $Bx = 0$, then $Ax = \rho^{-1}(B)x \Rightarrow Ax = \rho^{-1}(I_m)Bx = 0$.

Definition 8. Row-equivalent systems: *The systems of linear equations $Ax = b$ and $Cx = d$ are said to be row equivalent if their respective augmented matrices, $(A|b)$ and $(C|d)$ are row equivalent.*

Theorem 9. *Let $Ax = b$ and $Cx = d$ be two row equivalent linear systems. Then they have the same solution set.*

Proof: Let $E_1, E_2, \ldots, E_k$ be the elementary matrices such that $E_1E_2 \cdots E_k(A|b) = (C|d)$. Suppose y is a solution of $Ax = b$, i.e., $Ay = b$. Then $Cy = E_1E_2 \cdots E_kAy = E_1E_2 \cdots E_kb = d$. Similarly, we can prove that any solution of $Cx = d$ is a solution of $Ax = b$..

Definition 10. Row echelon form: *A form of a matrix satisfying the following properties is called row echelon matrix.*

1. *Every zero-row of A (row which has all its entries 0) occurs below every non-zero row (which has a non-zero entry);*

2. *Suppose the matrix has r nonzero rows (and remaining $m-r$ rows are zero). If the leading coefficient (the first non-zero entry) of i-th row $(1 \leq i \leq r)$ occurs in the k_i-th column, then $k_1 < k_2 < \cdots < k_r$,*

that is, the leading coefficient of each row after the first is positioned to the right of the leading coefficient of the previous row.

Definition 11. Row reduced echelon form: *A form of a matrix satisfying the following properties is called row reduced echelon form (in short RRE) or reduced echelon form:*

1. *The matrix is in row echelon form;*
2. *The leading coefficient of each row is 1;*
3. *All other elements in a column that contains a leading coefficient are zero.*

Remark 12. 1. *The process of computing row echelon form of a matrix by performing row operations is called* **Gaussian elimination**.

2. *The process of computing row-reduced echelon form of a matrix by applying row operations is called* **Gaussian-Jordan elimination**.
3. *Every matrix is row equivalent to a row-reduced echelon matrix. In fact, row-reduced echelon form of a matrix is unique.*

Example 13. *Find the RRE form of* $\begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix}$.

Solution:

$$\begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix} \underset{\sim}{R_3 \leftrightarrow R_4} \begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\sim}{R_1 \to \frac{1}{3} R_1} \begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{R_3 \to R_3 - 4R_1} \begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 2 & -4/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\sim}{R_2 \to \frac{1}{4} R_2} \begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 2 & -4/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\sim}{R_3 \to R_3 - 2R_2} \begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & -11/6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{R_3 \to \frac{-6}{11} R_3} \begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\sim}{R_1 \to R_1 - R_3, R_2 \to R_2 - \frac{1}{4} R_3} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 14. *Consider a system of m linear equations in n unknowns $Ax = 0$. Let R be the RRE form of A with r non-zero rows. The number of leading columns (column which contains a leading coefficient)* *is r. We call the variables associated with leading columns* **leading variables** *or* **dependent variables**. *The variables other than the dependent variables are called* **free variables** *or* **independent variables**. *Note that, either there is no equation in which the free variable appears or it appears with at least one another variable.*

For instance, if we consider a homogeneous system of linear equation $Ax = 0$, where A is as in Example 13. Then the columns 2,3 and 4 are leading columns and hence, x_2, x_3 and x_4 are leading variables or dependent variable, and x_1 is an independent (free) variable which is not appearing in any of the reduced equations.

Now consider a homogeneous system corresponding to the matrix $\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 4 \\ 1 & 2 & -1 & 2 \end{pmatrix}$. *Then the RRE form of A is* $\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *Here, x_1, x_3 are leading or dependent variables and x_2, x_4 are free or independent variables.*

Theorem 15. *If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations, $Ax = 0$, has a non trivial solution.*

Proof: Let R be the row-reduced echelon form of A. Then the systems $Ax = 0$ and $Rx = 0$ have the same solution set. If r is the number of non-zero rows in R, then $r \leq m$ so that $r < n$, equivalently, $n - r > 0$. Thus, there exists at least one independent variable (free variable) for the system $Rx = 0$, and hence $Rx = 0$ has a non-trivial solution. $\square$

Theorem 16. *Let $A \in M_{n\times n}(\mathbb{F})$. The matrix A is row equivalent to the $n \times n$ identity matrix if and only if the system of equations $Ax = 0$ has only the trivial solution.*

Proof: If A is row equivalent to the $n \times n$ identity matrix I_n, then $Ax = 0$ and $I_n x = 0$ have only the trivial solution. Conversely, suppose $Ax = 0$ has only the trivial solution and R is the RRE form of A. Let r be the number of non-zero rows of R. Then $r \leq n$. Note that $Rx = 0$ has only the trivial solution (as $Ax = 0$ and $Rx = 0$ are row equivalent) so that $r \geq n$. Hence, $r = n$ and $R = I_n$. $\square$