Lecture 23
Classification of Conics & Surfaces

Classification of Conics A conic is a curve in $\mathbb{R}^2$ which is represented by an equation of second degree in two variable, called quadratic curve. The general equation of such a conic (quadratic curve) is given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

where $a, b, h, g, f, c \in \mathbb{R}$ and $(a, b, h) \neq (0, 0, 0)$.

Then Equation (1) can be written as $(x, y)\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (2g, 2f)\begin{pmatrix} x \\ y \end{pmatrix} + c = 0$. Here, $H(X) = (x, y)\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = X^T AX$ is called the associated quadratic form of the conic (1), where $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$ is a symmetric matrix. Suppose λ_1, λ_2 are eigenvalues of A and P is an orthogonal matrix such that $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = P^T AP$. Then Equation (1) can be written as $(x, y)P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^T\begin{pmatrix} x \\ y \end{pmatrix} + +(2g, 2f)\begin{pmatrix} x \\ y \end{pmatrix} + c = 0$. Set $\begin{pmatrix} x' \\ y' \end{pmatrix} = P^T\begin{pmatrix} x \\ y \end{pmatrix}$. Then Equation (1) can be written as $\lambda_1 x'^2 + \lambda_2 y'^2 + 2g'x' + 2f'y' + c' = 0$. If $\lambda_1, \lambda_2 \neq 0$, then equation can be reduced to the following form

$$\lambda_1(x' + \alpha)^2 + \lambda_2(y' + \beta)^2 = \mu.$$

If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, the reduced equation is of the form $\lambda_2(y'_2 + \beta)^2 = \gamma x + \mu$ (similarly when $\lambda_1 \neq 0, \lambda_2 = 0$). If $\lambda_1 = \lambda_2 = 0$, then $2g'x' + 2f'y' + c' = 0$.

Proposition 1. *Consider the quadratic* $F(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$*, for* $a, b, c, g, f, h \in \mathbb{R}$*. If* $(a, b, h) \neq (0, 0, 0)$ *then the conic* $F(x, y) = 0$ *can be classified as follows.*

λ_1	λ_2	μ	conic
+ve	+ve	+ve	ellipse
+ve, -ve	-ve, +ve	non-zero	hyperbola
+ve	+ve	-ve	no real curve exists
+ve	+ve	0	single point
-ve	-ve	0	single point
+ve, -ve	-ve, +ve	0	pair of straight lines
0	±ve		parabola ($\gamma \neq 0$) or single line ($\gamma = 0 = \mu$) or pair of parallel lines ($\mu\lambda_2 > 0$) or two imaginary lines ($\mu\lambda_2 < 0$)
±ve	0		similar as above
0	0		single straight line

Example 2. *Identify the conic* $3x^2 - 2xy + 3y^2 - 8\sqrt{2}x + 10 = 0$

Solution: The matrix form is $(x,y)\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} + (-8\sqrt{2}, 0)\begin{pmatrix} x \\ y \end{pmatrix} + 10 = 0$. Eigenvalues of $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ are $2, 4$. The corresponding orthogonal matrix $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ such that $P^T AP = D$. Write $\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} x' \\ y' \end{pmatrix}$, we get $(x', y')\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} + (-8\sqrt{2}, 0)\begin{pmatrix} \frac{x'-y'}{\sqrt{2}} \\ \frac{x'+y'}{\sqrt{2}} \end{pmatrix} + 10 = 0$. By solving (expanding and making complete square), the reduced form is $2(x'-2)^2 + 4(y'+1)^2 = 2$, which represents an ellipse centered at $(2, -1)$.

Classification of Surfaces

A quadric surface is a surface in $\mathbb{R}^3$ described by a polynomial of degree 2 in three variables. A general equation of a surface is given by $F(x,y,z) = ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz + 2lx + 2my + 2nz + q$. The matrix form $F(x,y,z) = (x,y,z)\begin{pmatrix} a & h & g \\ g & b & f \\ g & f & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + (2l, 2m, 2n)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + q$. Let $A = \begin{pmatrix} a & h & g \\ g & b & f \\ g & f & c \end{pmatrix}$ and $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of A. Proceeding in a similar way as in the case of conics in $\mathbb{R}^2$, we get $F(x', y', z') = \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'2 + l'x + m'y + n'z + q'$. If $\lambda_1, \lambda_2, \lambda_3 \neq 0$, the equation can be reduced to the form $\lambda_1(x'+\alpha)^2 + \lambda_2(y'+\beta^2) + \lambda_3(z'+\gamma)^2 = \mu$. The classification of surfaces in $\mathbb{R}^3$ is as follows: If

λ_1	λ_2	λ_3	μ	conic
+ve	+ve	+ve	+ve	ellipsoid
+ve	+ve	-ve	+ve	hyperboloid of one sheet
+ve	-ve	-ve	+ve	hyperboloid of two sheet
+ve	+ve	+ve	0	single point
-ve	-ve	-ve	0	single point
+ve	+ve	-ve	0	cone
+ve	-ve	-ve	0	cone
+ve	+ve	0	+ve with coefficient of z is zero	elliptical cylinder
+ve	+ve	0	+ve with coefficient of z is non-zero	elliptical paraboloid
+ve	-ve	0	+ve with coefficient of z is zero	hyperbolic cylinder
+ve	-ve	0	+ve with coefficient of z is non-zero	hyperbolic paraboloid

two of the eigenvalues are zero, then the surface is either a parabolic cylinder or a pair planes or a singe plane.

Determine the following surface $F(x,y,z) = 0$, where $F(x,y,z) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz + 4x + 2y + 4z + 2$. Here $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $q = 2$. The eigenvalues of A are 4,1,1 and $P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$ such that $P^T AP = D$, where $D = diag(4,1,1)$. Hence, $F(x,y,z) = 0$ reduces to

$4\left(\frac{x+y+z}{\sqrt{3}}\right)^2+\left(\frac{x-y}{\sqrt{2}}\right)^2+\left(\frac{x+y-2z}{\sqrt{6}}\right)^2=-(4x+2y+4z+2)$. Further, we get $4\left(\frac{4(x+y+z)+5}{4\sqrt{3}}\right)^2+\left(\frac{x-y+1}{\sqrt{2}}\right)^2+\left(\frac{x+y-2z-1}{\sqrt{6}}\right)^2=9/12$. Equivalently, the surface can be written as $4(x'+5/4)^2+1(y'+1)^2+1(z'-1)^2=9/12$, where $x'=\frac{x+y+z}{\sqrt{3}}$, $y'=\frac{x-y}{\sqrt{2}}$, $z'=\frac{x+y-2z}{\sqrt{6}}$. Thus, the given equation describes an ellipsoid and the principal axes are $4(x+y+z)=-5$; $x-y=1$ and $x+y-2z=1$.