Lecture 2
System of Linear Equations

Definition 1. An equation of the form $a_1x_1 + \ldots + a_nx_n = b$, where $b, a_1, a_2, \ldots, a_n$ are constants, is called a **linear equation** in n unknowns. If the constants $a_1, \ldots, a_n$ and b are from a set X, the equation is called a linear equation over the set X.

Throughout this course, we deal with linear equations over the field $\mathbb{R}$ or $\mathbb{C}$.

System of linear equations: Let $\mathbb{F}$ be a field and $a_{ij}, b_j \in \mathbb{F}$, for $1 \leq i \leq m$, and $1 \leq j \leq n$. The following system is called a system of m linear equations in n unknowns over $\mathbb{F}$.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array} \tag{1}$$

If $b_j = 0$ for all $1 \leq j \leq m$, the system is called a **homogeneous system of linear equations**, otherwise it is called a **non-homogeneous system of linear equations**. The above system can be written as $\sum_{j=1}^{n} a_{ij}x_j = b_i$, for $1 \leq i \leq m$. An n-tuple $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n$ is called a **solution** of this system if it satisfies each of the equations of the system.

First we consider a system having only one equation

$$2x + 3y + 4z = 5.$$

Both $(1, 1, 0)$ and $(-1, 1, 1)$ satisfy this equation. In fact, for any real numbers x and y and we can find z by substituting the values of x and y in the equation. Geometrically, the collection of all the solutions of the equation $2x + 3y + 4z = 5$ is a plane in $\mathbb{R}^3$.

Now, we consider a system of two linear equations:

$$\begin{aligned} 2x + 3y + 4z &= 5 \\ x + y + z &= 2 \end{aligned}$$

A solution of this system is a solution of the first equation which is also a solution of the second equation. If A_i $(i = 1, 2)$ is the set of solutions of the i-th equation, then the set of solutions of the system is $A_1 \cap A_2$. Here, we know that for each i, A_i is a plane in $\mathbb{R}^3$. Thus, the solution of system is the intersection of two planes. In $\mathbb{R}^3$, the intersection of two planes is either an empty set (plane are parallel) or a line or a plane (the planes are identical). For this system, the solution set is $A_1 \cap A_2 = \{(1,1,0) + z(1,-2,1) : z \in \mathbb{R}\}$ which represents a line in $\mathbb{R}^3$. (Check it yourself!)

Remark 2. *1. A non-homogeneous system of 2 linear equations in 3 unknowns over $\mathbb{R}$ has either no solution or infinitely many solutions.*

2. A homogeneous system of 2 linear equations in 3 unknowns over $\mathbb{R}$ always has infinitely many solutions.

Now consider the following two systems of linear equations:

$$\begin{aligned} 2x + 3y + 4z &= 5 \\ x + y + z &= 2 \\ y + z &= 1 \end{aligned} \qquad \begin{aligned} 2x + 3y + 4z &= 5 \\ x + y + z &= 2 \\ x + 2y + 3z &= 2 \end{aligned} \qquad \begin{aligned} 2x + 3y + 4z &= 5 \\ x + y + z &= 2 \\ x + 2y + 3z &= 3 \end{aligned}$$

The first system has the unique solution, that is, $(1,1,0)$, the second system has no solution and the third system has more than one solution, in fact, infinitely many solutions.

Question 3. *When System (1) has no solution or a unique solution or infinitely many solutions?*

System (1) can be described by the following matrix equation:

$$Ax = b,$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in M_{m\times n}(\mathbb{F})$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in M_{n\times 1}(\mathbb{F})$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in M_{m\times 1}(\mathbb{F})$.

The matrix A is called the **coefficient matrix**, x is the matrix (or column) of unknowns and b is the matrix (or column) of constants.

The matrix $(A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \in M_{m\times(n+1)}(\mathbb{F})$ obtained by attaching the column b with A, is called the **augmented matrix** of the system.

Some properties of a system of linear equations

Let $Ax = b$ ($\sum_{j=0}^{n} a_{ij}x_j = b_i$ for $1 \leq i \leq m$) be a non-homogeneous system of linear equations and $Ax = 0$ ($\sum_{j=0}^{n} a_{ij}x_j = 0$ for $1 \leq i \leq m$) be the associated homogeneous system. Let S and S_h denote the solution sets of the systems $Ax = b$ and $Ax = 0$ respectively. The addition of two elements in $\mathbb{F}^n$ is given by

$$x + y = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

and the scalar multiplication is given by

$$\alpha x = \alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

Then we have the following statements.

P1: $x = (x_1, \ldots, x_n) \in S$ and $y = (y_1, \ldots, y_n) \in S_h \Rightarrow x + \alpha y \in S$ for all $\alpha \in \mathbb{R}$.

Proof: The i-th component of $A(x + \alpha y)$ is $\sum_{j=1}^{n} a_{ij}(x_j + \alpha y_j) = \sum_{j=1}^{n} a_{ij}x_j + \alpha \sum_{j=1}^{n} a_{ij}y_j = \sum_{j=1}^{n} a_{ij}x_j = b_i$ for $1 \leq i \leq m$. Therefore, $A(x + \alpha y) = b$ so that $x + \alpha y \in S$. $\square$

P2: Let $x \in S$ and $x + S_h := \{x + y \, : \, y \in S_h\}$. Then $S = x + S_h$.

Proof: P1$\Rightarrow x + S_h \subseteq S$. Also, for all $z \in S$, $z = x + (z - x) \in x + S_h \Rightarrow S \subseteq x + S_h$. $\square$

P3: If the system $Ax = b$ has more than one solution, then it has infinitely many solutions.

Proof: Let x, y be two solutions of $Ax = b$. Then it is easy to see that $\alpha x + (1 - \alpha)y$ is again a solution for each $\alpha \in \mathbb{R}$. $\square$

P4: If $Ax = 0$ has a non zero solution, then it has infinitely many solutions. (Do it yourself!)

Exercise 4. *Classify the following systems in the categories:*

1) The system has no solution 2) Exactly one solution 3) More than one solution

1. $x_1 + x_2 + x_3 = 3$, $x_1 + 2x_2 + 3x_3 = 6$, $x_2 + 2x_3 = 1$.

2. $x_1 + x_2 + x_3 = 3$, $x_1 + 2x_2 + 3x_3 = 6$, $x_1 + x_2 + 2x_3 = 4$.

3. $x_1 + x_2 + x_3 = 3$, $x_1 + 2x_2 + 3x_3 = 6$, $x_2 + 2x_3 = 3$.

Definition 5. An equation $d_1x_1 + d_2x_2 + \ldots + d_nx_n - e = 0$ is called a **linear combination** of the equations Eq_i if it can be written as $c_1\text{Eq}_1 + c_2\text{Eq}_2 + \cdots + c_n\text{Eq}_n$, where $\text{Eq}_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - b_i$ and $c_i \in \mathbb{F}$ for $1 \leq i \leq n$.

Remark 6. *If $(x_1, x_2, \ldots, x_n)$ is a solution of System (1), then it is a solution of $d_1x_1 + d_2x_2 + \ldots + d_nx_n - e = 0$. But converse need not be true. For instance, consider the following systems:*

$$\begin{aligned} x + y + z &= 1 \\ x + y &= 1 \\ x - z &= 1 \end{aligned} \qquad\qquad \begin{aligned} x + y + z &= 1 \\ 2x + y - z &= 2 \end{aligned}$$

Then latter one is obtained from former system. We see that $(-1, 3, -1)$ is solution of the latter one but not of the former one.

Definition 7. Two systems, say S_1 and S_2 of linear equations, are called equivalent if each equation of S_1 is a linear combination of the equations of S_2 and vice versa.

Theorem 8. The solution sets of equivalent systems of linear equations are identical.