Lecture 18

Orthogonal Projection & Shortest Distance

Definition 1. *Let $(V, \langle\,,\rangle)$ be an inner product space and W be a subspace of V. Let $v \in V$. The orthogonal projection $P_W(v)$ of v onto W is a vector in W such that*

$$\langle (v - P_W(v)), w\rangle = 0 \;\; \forall w \in W.$$

Theorem 2. Let W be a finite-dimensional subspace of an inner product space V with an orthonormal basis $\{w_1, \ldots, w_n\}$. The orthogonal projection of $v \in V$ onto W is $P_W(v) = \langle v, w_1\rangle w_1 + \ldots + \langle v, w_n\rangle w_n$.

Proof. Let $P_W(v) = w_v$. Note that $w_v \in W$ and $\{w_1, \ldots, w_n\}$ is a basis of W. Hence, $w_v = \langle w_v, w_1\rangle w_1 + \langle w_v, w_2\rangle w_2 + \cdots + \langle w_v, w_n\rangle w_n$. Further, $\langle v - w_v, w_i\rangle = 0 \Rightarrow \langle v, w_i\rangle - \langle w_v, w_i\rangle = 0 \Rightarrow \langle w_v, w_i\rangle = \langle v, w_i\rangle \forall i$. □

Remark 3. *1. $P_W(v) = w_v \Leftrightarrow ||v - w_v|| \leq ||v - w|| \,\forall\, w \in W$.*

2. Let v and u be two vectors in the inner product space V. Then orthogonal projection of u along v is $P_v(u) = \frac{<u,v>}{||v||^2} v$.

3. $P_W(v) \in W$ and $\langle v - P_W(v), w\rangle = 0$ for all $w \in W$, i.e., $v - P_W(v)$ is orthogonal to all the elements of W.

Definition 4. Let $(V, \langle\,,\rangle)$ be an inner product space and S be a non-empty subset of V. Then orthogonal complement of S, denoted by $S^\perp$, is defined as $S^\perp = \{v \in V \mid \langle v, s\rangle = 0 \,\forall s \in S\}$.

Definition 5. Let $(V, \langle\,,\rangle)$ be an inner product space and S_1 and S_2 be two subspaces of V. Then S_1 is perpendicular to S_2, $S_1 \perp S_2$, if $\langle s_1, s_2\rangle = 0$ for all $s_1 \in S_1$ and $s_2 \in S_2$.

Remark 6. *1. $V^\perp = \{0\}$.*

2. $\{0\}^\perp = V$.

3. Given any subset $W \subseteq V$, $W^\perp$ is a subspace of V.

4. $W \cap W^\perp = \{0\}$.

Theorem 7. Let $(V, \langle\,,\rangle)$ be an inner product space and and S_1 and S_2 are any two subsets of V. Then

1. $S \subseteq S^{\perp\perp}$.
2. if $S_1 \subseteq S_2$ then $S_2^\perp \subseteq S_1^\perp$.
3. if W is a finite-dimensional subspace of V, then $V = W \oplus W^\perp$.
4. if V is a finite-dimensional inner product space and W is a subspace of V, then $W = W^{\perp\perp}$.

Proof(i) Let $w \in S$ then $\langle w, v\rangle = 0$ for $v \in S^\perp$ which is equivalent to $w \in S^{\perp\perp}$. Thus $S \subseteq S^{\perp\perp}$

Proof(ii) Let $w \in S_2^\perp$ then $\langle w, v\rangle = 0$ for $v \in S_2$, but $S_1 \subseteq S_2$ hence $\langle w, v\rangle = 0$ for $v \in S_1$ this implies $w \in S_1^\perp$. Hence $S_2^\perp \subseteq S_1^\perp$.

Proof(iii) Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal basis of W. Then for any $v \in V$, $P_W(v) = \sum_{i=1}^{k} \langle v, v_i \rangle \frac{v_i}{||v_i||^2}$. Thus, for $v \in V$, $v = P_W(v) + (v - P_W(v)) \in W + W^{\perp}$. Further, $W \cap W^{\perp} = \{0\}$. Therefore, $V = W \oplus W^{\perp}$.

Proof(iv) By the above result, we have $V = W \oplus W^{\perp}$. Since $W^{\perp}$ is a subspace of V, $V = W^{\perp} \oplus W^{\perp\perp}$. Moreover, $W \subseteq W^{\perp\perp}$. Let $v \in W^{\perp\perp}$. Since V is finite dimensional, $\dim W + \dim W^{\perp} = \dim W^{\perp} + \dim W^{\perp\perp}$ so that $W = W^{\perp\perp}$.

Example 8. *Let* $V = M_n(\mathbb{R}), \mathbb{F} = \mathbb{R}$ *with inner product given by* $\langle A, B \rangle = tr(AB^T)$. *Let* W *be the space of diagonal matrices. Find* $W^{\perp}$.

Solution: A basis of W is given by $B = \{e_{11}, e_{22}, \ldots, e_{nn}\}$. Note that B is orthonormal.
$W^{\perp} = \{A \in M_n(\mathbb{R}) \mid tr(AB^T) = 0\ \forall A \in W\} = \{A \in M_n(\mathbb{R}) \mid tr(Ae_{ii}^T) = 0 \text{ for } i = 1, 2, \ldots, n\}$ $= \{A \in M_n(\mathbb{R}) \mid tr(Ae_{ii}) = 0 \text{ for } i = 1, 2, \ldots, n\} = \{A \in M_n(\mathbb{R}) \mid a_{ii} = 0 \text{ for } i = 1, 2, \ldots, n\}$. Thus $W^{\perp}$ is collection of matrices having diagonal entries zero.

Shortest distance of a point from a subspace

Definition 9. *Let* $(V, \langle\, , \rangle)$ *be an inner product space and* W *be its finite dimensional subspace. Then the shortest distance of a vector* $v \in V$ *is given by* $||v - P_W(v)||$.

Example 10. *Find the shortest distance of* $(1, 1)$ *from the line* $2y = x$.

Solution: Here $W = L(\{(2, 1)\})$. Note that $||(2, 1)||^2 = 5$ so that orthonormal basis of W is $\{(2, 1)/\sqrt{5}\}$. Thus, $P_W((1, 1)) = \langle (1, 1), (2, 1) \rangle \frac{(2,1)}{5} = \frac{3}{5}(2, 1)$. The shortest distance of $(1, 1)$ from the line $y = 2x$ is $||(1, 1) - \frac{3}{5}(2, 1)|| = ||(-1, 2)/5|| = \frac{1}{\sqrt{5}}$.