Lecture 13

Rank of a matrix & System of linear equations

Definition 1. Let $A \in M_{m\times n}(\mathbb{F})$. The **column space of** A is the linear span of columns of A, i.e., column space$(A) = L(\{(a_{11}, a_{21}, \ldots, a_{m1}), \ldots, (a_{1n},\ a_{2n},\ \ldots, a_{mn})\}) \subseteq \mathbb{F}^m$, and the **row space of** A is the linear span of the rows of A, i.e., the row space$(A) = L(\{(a_{11}, a_{12}, \ldots, a_{1n}), \ldots, (a_{1n}, a_{2n}, \ldots, a_{mn}\}) \subseteq \mathbb{F}^n$. The dimension of the column space of (A) is called the **column rank of** A and dimension of the row space of (A) is called the **row rank of** A.

Theorem 2. Let $A \in M_{m\times n}(\mathbb{F})$. Then Row rank$(A)$ = Column rank(A).

Proof: Let $R_1, R_2, \ldots, R_m$ be the rows of A. Then the i^{th} vector $R_i = (a_{i1}, a_{i2}, \ldots, a_{in})$. Suppose dimension of the row space of A is s and $\{v_1, v_2, \ldots, v_s\}$ is a basis of the row space of A. Then

$$\begin{aligned} R_1 &= c_{11}v_1 + c_{12}v_2 + \ldots + c_{1s}v_s \\ R_2 &= c_{21}v_1 + c_{22}v_2 + \ldots + c_{2s}v_s \\ &\vdots \\ R_m &= c_{m1}v_1 + c_{m2}v_2 + \ldots + c_{ms}v_s \end{aligned}$$

Let $v_j = (b_{j1}, b_{j2}, \ldots, b_{jn})$ for $1 \leq j \leq s$. Then $a_{1i} = c_{11}b_{1i} + c_{12}b_{2i} + c_{1s}b_{si}$, $a_{2i} = c_{21}b_{1i} + c_{22}b_{2i} + c_{2s}b_{si}$, ..., $a_{mi} = c_{m1}b_{1i} + c_{m2}b_{2i} + c_{ms}b_{si}$. This implies, $(a_{1i}, a_{2i}, \ldots, a_{mi}) = b_{1i}(c_{11}, c_{21}, \ldots, c_{m1}) + \ldots + b_{si}(c_{1s}, c_{2s}, c_{ms})$. Thus, each column vector is a linear combination of s vectors $\{(c_{11}, c_{21}, \ldots, c_{m1}), (c_{12}, c_{22}, \ldots, c_{m2}), \ldots, (c_{1s}, c_{2s}, \ldots, c_{ms})\}$. Therefore, dim(column space) $\leq s =$ dim(row space). Similarly, we can show that dim(row space) $\leq s =$ dim(column space). □

Definition 3. The rank of a matrix A is the dimension of row space of A (or the dimension of column space of A).

Definition 4. *The nullity of a matrix A is the dimension of the solution space of $Ax = 0$.*

Theorem 5 (Rank-Nullity Theorem for a Matrix)**.** *Let $A \in M_{m\times n}(\mathbb{R})$. Then*

$$rank(A) + nullity(A) = number\ of\ columns\ of\ A.$$

Proof. Recall that there is a one to one correspondence between $L(\mathbb{R}^n, \mathbb{R}^m)$ and $M_{m\times n}(\mathbb{R})$. Consider the map ϕ such that $T \mapsto [T]_B^{B'}$, where B and B' be the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then ϕ is linear one-one and onto. For onto, given a matrix A, take the linear transformation T_A given by $T_A(x) = Ax$. □

Remark 6. *1. The rank of a matrix A is same as the number of non-zero rows in its RRE form.*

Proof. Let the number of non zero rows in the RRE form of A is r. Observe that a row obtained by applying an elementary row operation is nothing but a linear combination of rows of the matrix, and the rows in RRE form are LI. Therefore, the dimension of row space or rank of A is r. □

Determinantal-Rank of a matrix

Let $A \in M_{m\times n}(\mathbb{R})$. Then A has determinantal-rank r if
1. every $k \times k$ submatrix of A has zero determinant, where $k > r$;
2.there exist an $r \times r$ submatrix with non-zero determinant.

Theorem 7. *Rank*(A)=*Determinantal Rank*(A).

Proof. Let rank(A) = l and determinantal-rank(A) = r. We show that $r = l$. Since determinantal-rank$(A) = r$, there exists an $r \times r$ submatrix R with non-zero determinant so that rank$(R) = r$, equivalently, all rows of R are linearly independent. Then the corresponding r rows of matrix A are LI. Therefore, $r \leq rank(A)$.

Let B be a submatrix of A consisting of linearly independent rows of A. Let $rank(A) = l$. Then order of B is $l \times n$ and $rank(B)$ is l. Hence, B has l linearly independent columns. Consider an $l \times l$ submatrix B' of B (also a submatrix of A) having those l linearly independent columnsof B. Then $rank(B') = l$ so that $|B'| \neq 0$. Therefore, $l \leq r$. □

Application of rank in system of linear equations

First we recall a result on system of linear equation:

Theorem 8. *Let $Ax = b$ be a non-homogeneous system of linear equations, and $Ax = 0$ be the associated homogeneous system. If $Ax = b$ is consistent and x_0 is a particular solution of $Ax = b$, then any solution of $Ax = b$ can be written as $x = x_h + x_0$, where x_h is a solution of $Ax = 0$.*

Let $A \in M_{m\times n}(\mathbb{R})$ and Rank$(A) = r$. Suppose $Ax = b$ is a non-homogeneous system of linear equations, and $Ax = 0$ is the associated homogeneous system. Then

1. $Ax = b$ is consistent if and only if Rank$(A \mid b) = r$.

   Solution: If $Ax = b$ is consistent, then $b \in$ Column Space(A) so that Rank$(A \mid b) = r$. Similarly, the other way.

2. Let $Ax = b$ be consistent. Then the solution is unique if and only if $r = n$.

   Solution: Let $Ax = b$ have a unique solution. Then $Ax = 0$ has a unique solution, *i.e.*, the zero solution. This implies $nullity(A) = 0$. Then by rank-nullity theorem, we have $n = rank(A)$ and vice-versa.

3. If $r = m$, then $Ax = b$ always has a solution for every $b \in \mathbb{R}^m$.

   Solution: If $r = m$, then the column space is $\mathbb{R}^m$. Thus each vector in $\mathbb{R}^m$ is a linear combination of columns of A. Hence, $Ax = b$ has a solution for all $b \in \mathbb{R}^m$.

4. If $r = m = n$ then $Ax = b$ always has a unique solution for all b and further $Ax = 0$ has only zero solution.

   Solution: Since $r = m$, the column space is $\mathbb{R}^m$. Therefore, $Ax = b$ always has a solution for all b. Further, $nullity(A) = 0$. Thus, $Ax = 0$ has only zero solution and hence, $Ax = b$ always has a unique solution all b.

5. If $r = m < n$, for any $b \in \mathbb{R}^m$, $Ax = b$ as well as $Ax = 0$ have infinitely many solutions.

   Solution: Since $r = m$, $Ax = b$ has a solution for all $b \in \mathbb{R}^m$. Note that, nullity$(A) = (n - r) > 0$. Therefore, $Ax = 0$ has infinitely many solutions and hence, $Ax = b$ has infinitely many solutions.

6. In case (i) $r < m = n$, (ii) $r < m < n$ and (iii) $r < n < m$, if $Ax = b$ has a solution then there are infinitely many solutions.

   Solution: Note that nullity$(A) = (n - r) > 0$. Hence $Ax = 0$ has infinitely many solutions. Now if $Ax = b$ has a solutions then it has infinitely many solutions.

7. If $r = n < m$, then $Ax = 0$ has only zero solution and if $Ax = b$ has a solution, the solution is unique.

   Solution: In this case, nullity$(A) = 0$, implies $Ax = 0$ has only trivial solution. If $Ax = b$ has a solution, then it is unique.

Example 9. *Let $T : P_2(\mathbb{R}) \Rightarrow \mathbb{R}^2$ given by $T(p(x)) = (p(0), p(1))$. Find rank(T), nullity(T), basis of ker(T) and basis range(T).*

Solution: *Let $B = \{1, x, x^2\}$ and $B' = \{e_1, e_2\}$. Then*

$$[T]_B^{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = A.$$

$RRE(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Thus, $Rank(T) = Rank(A) = 2$ so that range of T is $\mathbb{R}^2$ and its basis is $\{e_1, e_2\}$. Further, $nullity(T) = nullity(A) = nullity(RRE(A)) = 3 - 2 = 1$. The solution space of $Ay = 0$ is $\{(0, a, -a) \mid a \in \mathbb{R}\}$. Note that $y = [v]_B$, therefore, the ker$(T) = \{ax + (-a)x^2 \mid a \in \mathbb{R}\}$. Hence, basis of kernel T is $\{x - x^2\}$.