Lecture 12

Matrix Representation of a Linear Transformation & Similar Matrices

Definition 1. Let $B_1 = \{v_1, v_2, \ldots, v_n\}$ and $B_2 = \{u_1, \ldots, u_n\}$ be ordered bases of a vector space V over $\mathbb{F}$. Then the matrix $P_{B_1 \mapsto B_2}$ having the i-th $(1 \leq i \leq n)$ column as the coordinate vector of v_i with respect to the basis B_2, that is,

$$P_{B_1 \mapsto B_2} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix},$$

where $[v_i]_{B_2} = (p_{1i}, p_{2i}, \ldots, p_{ni})^T$, is called the transition matrix from the basis B_1 to the basis B_2.

Theorem 2. Let $B_1 = \{v_1, \ldots, v_n\}$ and $B_2 = \{u_1, \ldots, u_n\}$ be ordered bases of a vector space V over $\mathbb{F}$. If v is a vector in V, then $[v]_{B_2} = P_{B_1 \mapsto B_2}[v]_{B_1}$, where $P_{B_1 \mapsto B_2}$ is the transition matrix from B_1 to B_2.

Proof: Let $[v]_{B_1} = (a_1, a_2, \ldots, a_n)^T$. Then $v = a_1 v_1 + \ldots + a_n v_n$. We know $[v_i]_{B_2} = P_i$, where $i = 1, \ldots, n$ and P_i is the i-th column of $P_{B_1 \mapsto B_2}$. Thus, $v_1 = p_{11}u_1 + \ldots + p_{n1}u_n$, $v_2 = p_{12}u_1 + \ldots + p_{n2}u_n$, $\ldots$, $v_n = p_{1n}u_1 + \ldots + p_{nn}u_n$, where p_{ij} is the (i, j)-th entry of $P_{B_1 \mapsto B_2}$. Putting these value in (1), we get $v = (a_1 p_{11} + a_2 p_{12} + \cdots + a_n p_{p1n})u_1 + \cdots + (a_1 p_{n1} + a_2 p_{n1} + \cdots + a_n p_{nn})u_n$. Therefore, $[v]_{B_2} = P_{B_1 \mapsto B_2}[v]_{B_1}$. $\square$

Theorem 3. A transition matrix is invertible.

Proof: Let V be a vector space over $\mathbb{F}$ and B_1 and B_2 are bases of V. For $v \in V$, $P_{B_1 \mapsto B_2}[v]_{B_1} = [v]_{B_2}$. Let $P_{B_2 \mapsto B_1}$ be the transition matrix from basis B_2 to B_1. Then $P_{B_1 \mapsto B_2} P_{B_1 \mapsto B_2}[v]_{B_1} = P_{B_1 \mapsto B_2}[v]_{B_2} = [v]_{B_1}$. Thus, $P_{B_1 \mapsto B_2}$ is invertible (using Exercise: If a square matrix has a left inverse, then it is invertible).

Example 4. *Let $B_1 = \{(1,1), (1,-1)\}$ and $B' = \{(1,2), (2,1)\}$ be bases of $\mathbb{R}^2(\mathbb{R})$. Then*

$$P_{B_1 \mapsto B_2} = \begin{pmatrix} \frac{1}{3} & -1 \\ \frac{1}{3} & 1 \end{pmatrix} \text{ and } P_{B_2 \mapsto B_1} = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}.$$

Let $(x, y) \in \mathbb{R}^2$. Then $[(x, y)]_{B_1} = (\frac{x+y}{2}, \frac{x-y}{2})^t$ and $[(x, y)]_{B_2} = (\frac{2y-x}{3}, \frac{2x-y}{3})^t$. Verify that $[(x, y)]_{B_2} = P_{B_1 \mapsto B_2}[(x, y)]_{B_1}$ and $[(x, y)]_{B_1} = P_{B_2 \mapsto B_1}[(x, y)]_{B_2}$. Also, $P_{B_1 \mapsto B_2}$ is inverse of $P_{B_2 \mapsto B_1}$.

Matrix representation of a linear transformation: Let V and W be vector spaces over $\mathbb{F}$ with ordered bases $B_V = \{v_1, v_2, \ldots, v_m\}$ and $B_W = \{w_1, w_2, \ldots, w_n\}$ respectively. Let $T : V \to W$ be a linear transformation. Then

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i \text{ for } (1 \leq j \leq m),$$

where $\alpha_{ij} \in \mathbb{F}$ and we get an $n \times m$ matrix M_T given by

$$M_T = \alpha_{ij},$$

that is, the i-th column of M_T is the coordinate vector $[T(v_i)]_{B_W}$. The matrix M_T is called the **matrix representation of T with respect to the bases B_V and B_W**. Since the coordinate vectors are unique, M_T is also unique. We also denote the matrix representation of T with respect to B_V and B_W by $[T]_{B_V}^{B_W}$. If T is an operator ($T : V \to V$) and both the bases are identical, then we simply write $[T]_B$.

Theorem 5. Let $v \in V$. Then $[T(v)]_{B_W} = [T]_{B_V}^{B_W} [v]_{B_V}$.

Example 6. *1. Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x, y, z) = (2x + z, y + 3z)$ with $B = \{(1, 1, 0), (1, 0, 1), (1, 1, 1)\}$ and $B' = \{(2, 3), (3, 2)\}$. Then*

$$\begin{aligned}
T(1,1,0) &= (2,1) = \frac{-1}{5}(2,3) + \frac{4}{5}(3,2) \\
T(1,0,1) &= (3,3) = \frac{3}{5}(2,3) + \frac{3}{5}(3,2) \\
T(1,1,1) &= (3,4) = \frac{6}{5}(2,3) + \frac{1}{5}(3,2)
\end{aligned}$$

Thus, the matrix representation of T with respect to B and B' is

$$\begin{pmatrix} -\frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ \frac{4}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}.$$

2. Let $D : P_2 \longrightarrow P_1$ be the differential operator. Find the matrix representations of D from $B = \{1, x, x^2\}$ to $B' = \{1, 1 + x\}$.

$$\begin{aligned}
D(1) &= 0 = 0(1) + 0(1+x) \\
D(x) &= 1 = 1(1) + 0(1+x) \\
D(x^2) &= 2x = -2(1) + 2(1+x) \\
[D]_B^{B'} &= \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}.
\end{aligned}$$

Definition 7. Let $A, B \in M_n(\mathbb{F})$. Then A and B are said to be similar if there exist an invertible matrix $P \in M_n(\mathbb{F})$ such that $A = P^{-1}BP$.

Theorem 8. Let $V(\mathbb{F})$ be a vector space with ordered bases B and B'. Let T be a linear operator on V (that is, $T : V \to V$). If $[T]_B = A$ and $[T]_{B'} = A'$, then $A' = P^{-1}AP$, where P is the transition matrix from B' to B.

Proof: Let $v \in V$. Then $[T(v)]_B = A[v]_B$ and $[T(v)]_{B'} = A'[v]_{B'}$. We know that $P[v]_{B'} = [v]_B$

so that $P([T(v)]_{B'}) = [T(v)]_B = A[v]_B$. Also, $P([T(v)]_{B'}) = P(A'[v]_{B'}) = P(A'P^{-1}[v]_B)$. Therefore, $A[v]_B = PA'P^{-1}[v]_B \ \forall [v]_B$. Hence, $A = PA'P^{-1}$ or $A' = P^{-1}AP$. □

Theorem 9. Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be vector spaces. Suppose B_1, B_1' are bases for V and B_2, B_2' are ordered bases for W. Then for any linear map $T : V \longrightarrow W$,

$$[T]_{B_1'}^{B_2'} = Q[T]_{B_1}^{B_2}P,$$

where $P = P_{B_1' \to B_1}$ and $Q = Q_{B_2 \to B_2'}$.

Example 10. *Consider Example 6 (1), let* $B_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$, $B_1' = \{(1,1,0),(1,0,1),(1,1,1)\}$ *and* $B_2 = \{(1,0),(0,1)\}$. *and* $B_2' = \{(2,3),(3,2)\}$. *Then*

$$[T]_{B_1}^{B_2} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix} \text{ and } [T]_{B_1'}^{B_2'} = \begin{pmatrix} -\frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ \frac{4}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}.$$

The transition matrices

$$P = P_{B_1' \mapsto B_1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } Q = Q_{B_2 \mapsto B_2'} = \begin{pmatrix} \frac{-2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{-2}{5} \end{pmatrix}.$$

Verify that $[T]_{B_1'}^{B_2'} = Q[T]_{B_1}^{B_2}P$.