Lecture 4

Basis & Dimension of Direct Sum of Subspaces

Theorem 1. *If W is a subspace of a finite dimensional vector space V, every linearly independent subset of W is finite and it is a part of a basis for W.*

We say that W is a proper subspace of a vector space V if $W \neq \{0\}$ and $W \neq V$.

Theorem 2. *If W is a proper subspace of a finite-dimensional vector space V, then W is finite-dimensional and* $\dim W < \dim V$.

Proof: Since W is not the zero space, then $\exists\, w \in W$ such that $w \neq 0$. There is a basis B of W containing w. Note that B can have at most n vectors as V is n dimensional. Hence W is finite-dimensional, and $\dim W \leq \dim V$. Since W is a proper subspace, there is a vector v in V which is not in W. Adjoining v to B, we obtain a linearly independent subset of V. Thus $\dim W < \dim V$.

Theorem 3. If W_1 and W_2 are two subspaces of a finite dimensional vector space V, then $W_1 + W_2$ is finite dimensional and $\dim(W_1 + W_2) = \dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$.

Proof: Since $W_1 \cap W_2$ is a subspace of W_1 as well as of W_2, it is finite dimensional. If $B_0 = \{w_1, \ldots, w_k\}$ is a basis of $W_1 \cap W_2$, then B_0 can be extended to a basis for W_1 as well as of W_2. Let $B_1 = \{w_1, \ldots, w_k, v_1, \ldots, v_l\}$ and $B_2 = \{w_1, \ldots, w_k, u_1, \ldots, u_m\}$ be bases of W_1 and W_2 respectively. We claim that the set $B = B_0 \cup B_1 \cup B_2 = \{w_1, \ldots, w_k, v_1, \ldots, v_l, u_1, \ldots, u_m\}$ forms a basis of the subspace $W_1 + W_2$. Clearly, $L(B) = W_1 + W_2$. We need to show that B is a linearly independent set. Let $\sum_{i=1}^{k} \alpha_i w_i + \sum_{j=1}^{l} \beta_j v_j + \sum_{r=1}^{m} \gamma_r u_r = 0$, where $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$. Then

$$\sum_{i=1}^{k} \alpha_i w_i + \sum_{j=1}^{l} \beta_j v_j = -\sum_{r=1}^{m} \gamma_r u_r$$

so that $-\sum_{k=1}^{m} \gamma_k u_k = W_1 \cap W_2$ (as RHS is in W_2 and LHS is in W_1). Therefore,

$$-\sum_{r=1}^{m} \gamma_r u_r = \sum_{i=1}^{k} \delta_i w_i$$

so that $\sum_{r=1}^{m} \gamma_r u_r + \sum_{i=1}^{k} \delta_i w_i = 0$. But $\{w_1, \ldots, w_k, u_1, \ldots, u_m\}$ is a basis of W_2, therefore $\gamma_r = 0$ for $1 \leq r \leq m$. This further implies that $\alpha_i = \beta_j = 0$. Thus, the set B forms a basis for $W_1 + W_2$. □

Corollary 4. *Let W_1, W_2 be subspaces of V. Then*

$$\dim W_1 + \dim W_2 - \dim V \leq \dim(W_1 \cap W_2) \leq \min\{\dim W_1, \dim W_2\}.$$

Definition 5. *Let W_1 and W_2 be subspaces of a vector space V. The vector space V is called the* **direct sum** *of W_1 and W_2, denoted as $W_1 \oplus W_2$, if every element $v \in V$ can be uniquely represented as $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$.*

Theorem 6. *A vector space $V(\mathbb{F})$ is the direct sum of its subspaces W_1 and W_2 if and only if $V = W_1 + W_2$, and $W_1 \cap W_2 = \{0\}$.*

Proof: Let $V = W_1 \oplus W_2$. Since every elements $v \in V$, $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. Thus, $W_1 + W_2 = V$. Let $x \in W_1 \cap W_2$. Then $x = x + 0$ and $x = 0 + x$. But x must have a unique representation, therefore $x = 0$.

Conversely, let $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Suppose $v \in V$ has more than one representation, *i.e.*, $v = w_1 + w_2 = w_1' + w_2'$. This implies $w_1 - w_1' = w_2 - w_2' \in W_1 \cap W_2 = \{0\}$. Thus $w_1 = w_1'$ and $w_2 = w_2'$. This follows the proof. □

Corollary 7. $\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2$.

Example 8. *Let $V = \mathbb{R}^2(\mathbb{R})$ and $W_1 = \{(x, 2x) \mid x \in \mathbb{R}\}$, $W_2 = \{(x, 3x) \mid x \in \mathbb{R}\}$ be subspaces of V. Then $V = W_1 \oplus W_2$.*

Note that, $(x, y) = (3x - y, 2(3x - y)) + (y - 2x, 3(y - 2x))$. Let $(x, y) \in W_1 \cap W_2$ then $(x, y) = (a, 2a) = (b, 3b)$ for some $a, b \in \mathbb{R}$. Then $(x, y) = (0, 0)$ so that $W_1 \cap W_2 = \{0\}$.