Practice Set - II

1. Let X be a discrete random variable taking values in $\mathcal{S} = \{-3, -2, -1, 0, 1, 2, 3\}$ such that $P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$ and $P(X < 0) = P(X = 0) = P(X > 0)$. Find the distribution function of X.

2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by
$$f(x) = \begin{cases} Cxe^{-x/2}, & x > 2, \\ 0, & x \leq 0, \end{cases}$$
What is the probability that the system functions for at least 5 months?

3. A circular board with radius 1 is sectioned into n concentric discs with radii $1/n, 2/n, \ldots, 1$. A dart is thrown, randomly inside the circle. If it hits the ring between the circles with radii i/n and $(i+1)/n$ for $i = 0, \ldots, n-1$, $(n-i)$ monetary units are won. Let X be the random variable that denotes the amount of money won. Find the probability mass function of the random variable X.

4. A player takes out, simultaneously and randomly, two balls from a box that contains 8 white balls, 5 black balls and 3 blue balls. Suppose that the player wins Rs. 2 for each black ball selected and looses Rs. 1 for each white ball selected. Let X be a random variable that denotes the players fortune. Find the probability mass function of the random variable X. Also find $P(X^2 > 3)$.

5. The distribution function of a random variable X is given by
$$F(x) = \begin{cases} 0, & x < -2, \\ 1/3, & -2 \leq x < 0, \\ 1/2, & 0 \leq x < 5, \\ 1/2 + (x-5)^2/2, & 5 \leq x < 6, \\ 1, & x \geq 6, \end{cases}$$
Find $P(2- \leq X < 5)$, $P(0 < X < 5.5)$ and $P(1.5 < X \leq 5.5 | X > 2)$.

6. Find the value of α and k so that F given by
$$F(x) = \begin{cases} 0, & x \leq 0, \\ \alpha + ke^{-x^2/2}, & \text{x¿0}, \end{cases}$$
is distribution function of a continuos random variable.

7. Let
$$F(x) = \begin{cases} 0, & x < 0, \\ (x+2)/8, & 0 \leq x < 1, \\ (x^2+2)/8, & 1 \leq x < 2, \\ (2x+c)/8 & 2 \leq x \leq 3, \\ 1, & x > 3, \end{cases}$$

Find the value of c such that F is a distribution function.

8. Let X be a random variable such that $P(X > 1/2) = 7/8$ and its pdf is:

$$f(x) = \begin{cases} ax, & 0 < x < 1, \\ b - x, & 1 < x \leq < 2, \\ 0, & \text{otherwise} \end{cases}$$

Determine a, b and the distribution function of X.

9. A battery cell is labeled as good if it works for at least 300 days in a clock, otherwise it is labeled as bad. Three manufacturers, A,B and C make cells with probability of making good cells as 0.95, 0.90 and 0.80 respectively. Three identical clocks are selected and cells made by A,B and C are used in clock numbers 1, 2 and 3 respectively. Let X be the total number of clocks working after 300 days. Find the probability mass function of X and plot the corresponding distribution function.

10. An urn contains n cards numbered $1, 2, \cdots, n$. Let X be the least number on the m cards drawn randomly without replacement from the urn. Find probability distribution of random variable X. Compute $P(X \geq 3/2)$.

11. An insurance company writes a policy to the effect that an amount of money A must be paid if some event E occurs within a year. If the company estimates that E will occur within a year with probability p, what should it charge the customer so that its expected profit will be 10 percent of A?

12. A total of 4 buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

    (a) Which of $E(X)$ or $E(Y)$ do you think is larger? Why?

    (b) Compute $E(X)$ and $E(Y)$.

13. Suppose that two teams play a series of games that end when one of them has won i games. Suppose that each game played is, independently, won by team A with probability p. Find the expected number of games that are played when $i = 2$. Also show that this number is maximized when $p = 1/2$.

14. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2, & 0 \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

If $E(X) = 3/5$, find a, b.

15. The lifetime in hours of electronic tubes is a random variable having a probability density function given by

$$f(x) = a^2 x e^{-ax}, \quad x \geq 0.$$

Compute the expected lifetime of such a tube.

16. (a) If $E(X) = 2$ and $E(X^2) = 8$, calculate (a) $E((2+4X)^2))$ and (b) $E(X^2+(X+1)^2)$.
   (b) Ten balls are randomly chosen from an urn containing 17 white and 23 black balls. Let X denote the number of white balls chosen. Compute $E(X)$.

17. If X is a continuous random variable having distribution function F , then its median is defined as that value of m for which $F(m) = 1/2$. Find the median of the random variables with density function
   (a) $f(x) = e^{-x}$, $x \geq 0$,
   (b) $f(x) = 1$, $0 \leq x \leq 1$.

18. A community consists of 100 married couples. If during a given year 50 of the members of the community die, what is the expected number of marriages that remain intact? Assume that the set of people who die is equally likely to be any of the $\binom{200}{50}$ groups of size 50.

19. Let $p_i = PX = i$ and suppose that $p_1 + p_2 + p_3 = 1$. If $E(X) = 2$, what values of p_1, p_2, p_3 (a) maximize and (b) minimize $Var(X)$?

20. Argue that for any random variable X, $E(X^2) \geq (E(X))^2$. When does one have equality?

21. A random variable Z , which represents the weight (in ounces) of an article, has density function given by $f(z)$,

$$f(z) = \begin{cases} (z-8), & 8 \leq z \leq 9 \\ (10-z), & 9 \leq z \leq 10 \\ 0, & otherwise \end{cases}$$

   (a) Calculate the mean and variance of the random variable Z.
   (b) The manufacturer sells the article for a fixed price of Rs. 2. He guarantees to refund the purchase money to any customer who finds the weight of his article to be less than 8.25oz. His cost of production is related to the weight of the article by the relation $x/15 + .35$. Find the expected profit per article.

22. A machine makes a product that is screened (inspected 100 percent) before being shipped. The measuring instrument is such that it is difficult to read between $1\frac{1}{3}$ (coded data). After the screening process takes place, the measured dimension has density

$$f(z) = \begin{cases} kz^2, & 0 \leq z \leq 1 \\ 1, & 1 < z \leq 1\frac{1}{3} \\ 0, & otherwise \end{cases}$$

(a) Find the value of k.

(b) What fraction of the items will fall outside the twilight zone (fall between 0 and 1)?

(c) Find the mean and variance of this random variable.