Indian Institute of Information Technology, Allahabad
M. Tech. 1st Semester
Statistical Foundations for Data Science (DSF)
C1 Review Test

Date: February 22, 2022 (3:00 PM - 4:00 PM) Total Marks: 25

Important Instructions:

1. Attempt all questions. There is no credit for a solution if the appropriate work is not shown, even if the answer is correct. All the notations are standard and same as used in the lecture notes.
2. Write down your name, enrolment number on a blank paper. Write the solutions clearly with all the steps in detail.
3. Submit the solution in PDF format through Google Classroom. **Name the PDF as your C1_enrolment number**. We will not accept the solution through emails.
4. 10 extra minutes are given for uploading the answer sheet on google classroom. Any submission after 4:10 PM will lead to penalty.

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1. There are two children in a family. Let E_1 be the event that the first child is a boy, let E_2 be the event that the second child is a boy and let E_3 be the event that both the children are of same gender. Are E_1, E_2 and E_3 [6]

   (a) pairwise independent?

   (b) independent?

   Solution: Here, $E_1 = \{BB, BG\} \Rightarrow P(E_1) = 1/2.$ [1/2]
   $E_2 = \{BB, GB\} \Rightarrow P(E_2) = 1/2.$ [1/2]
   $E_3 = \{BB, GG\} \Rightarrow P(E_3) = 1/2.$ [1/2]
   $E_1 \cap E_2 = E_2 \cap E_3 = E_1 \cap E_3 = E_1 \cap E_2 \cap E_3 = \{BB\}$ [1/2]
   Since
   $P(E_1 \cap E_2) = P(E_1)P(E_2)$ [1]
   $P(E_2 \cap E_3) = P(E_2)P(E_3)$ [1]
   $P(E_2 \cap E_3) = P(E_2)P(E_3)$ [1]
   Therefore E_1, E_2 and E_3 are pairwise independent.
   But, $P(E_1 \cap E_2 \cap E_3) \neq P(E_1)P(E_2)P(E_3).$ [1]
   Therefore, E_1, E_2 and E_3 are not independent.

2. Suppose Maria is playing a game where she is given two questions and she is to decide which question is to be answered first. The probability of answering question 1 correctly is 0.7 and Maria will receive a prize of Rs. 200 for the correct answer. The probability of answering question 2 correctly is 0.5 and Maria will receive a price of Rs. 400 for the correct answer. The game gets terminated if the first question is answered incorrectly. Maria will be allowed to attempt the second question if the first question is answered

correctly. Which question should be answered first to maximize the expected value of the total prize money received? [8]

Solution: Let X denote the total prize money received.

(a) Answer question 1 first: The p.m.f. of X is

$$p_X(x) = \begin{cases} 0.3, & x = 0 \\ (0.7) * (0.5) = 0.35, & x = 200 \\ (0.7) * (0.5) = 0.35, & x = 600 \end{cases}$$ [1+1+1]

So we have $E(X) = 0.35 * 200 + 0.35 * 600 = 70 + 210 = 280.$ [1]

(b) Answer question 2 first: The p.m.f. of X is

$$p_X(x) = \begin{cases} 0.5, & x = 0 \\ (0.5) * (0.3) = 0.15, & x = 400 \\ (0.5) * (0.7) = 0.35, & x = 600 \end{cases}$$ [1+1+1]

So we have $E(X) = 0.15 * 400 + 0.35 * 600 = 60 + 210 = 270.$ [1]

Thus, it is preferable to attempt the easier question 1 first.

3. Let X have the following PDF

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x < 2. \end{cases}$$

Then find [8]

(a) The CDF of X.
(b) $P(\frac{1}{6} < X \leq \frac{7}{4})$.
(c) $P(\frac{4}{3} < X < 3 | X \geq 1)$
(d) MGF of X.

So;ution:

(a) The CDF is given by

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{2}, & 0 \leq x < 1 \\ 2x - \frac{x^2}{2} - 1, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$ [3]

(b) $P(\frac{1}{6} < X \leq \frac{7}{4}) = \int_{\frac{1}{6}}^{1} x dx + \int_{1}^{\frac{7}{4}} (2 - x) dx = 275/288$ [1]

(c) $P(\frac{4}{3} < X < 3 | X \geq 1) = \frac{P(4/3 < X < 2)}{P(X \geq 1)} = \frac{2/9}{1/2} = 4/9.$ [2]

(d) The MGF is given by $E(e^{tX}) = \int_0^1 xe^{tx}dx + \int_1^2 (2-x)e^{tx}dx = \frac{\left(e^t-1\right)^2}{t^2}, t > 0.$ [2]

4. If a machine produces a defective item with probability 0.2, independent of any other item. What is the probability that in a sample of 5 items, at most one item will be defective? [3]
Solution: Let X be the number of defectives then this problem can be modeled with binomial distribution $\text{Bin}(5, 0.3)$. [1]
The required probability is given by
$P(X \leq 1) = P(X = 0) + P(X = 1) = \binom{5}{0}(0.2)^0(0.8)^5 + \binom{5}{1}(0.2)^1(0.8)^4 = 0.73728.$[1+1]