Probability and Statistics
Problem Set V

Note: Normal Approximation to Binomial Bin(n, p) is good if $np(1-p) \geq 10$. Similarly Normal Approximation to Poisson $P(\lambda)$ is good if $\lambda \geq 10$.

Table of continuity correction:

Discrete	Continuous
$P(X=a)$	$P(a-0.5 < X < a+0.5)$
$P(X>a)$	$P(X > a+0.5)$
$P(X \leq a)$	$P(X < a+0.5)$
$P(X<a)$	$P(X < a-0.5)$
$P(X \geq a)$	$P(X > a-0.5)$

1. Let $X \sim \mathrm{N}(\mu, \sigma^2)$ be a normal random variable. Then show that the random variable $Y = (X-\mu)^2/\sigma^2$ is distributed as gamma distribution $G(1/2, 1/2)$. Let us now consider n independent Normal random variables $X_i \sim \mathrm{N}(\mu_i, \sigma_i^2)$, $i = 1, 2, ..., n$, and define the new variable
$$\chi_n^2 = \sum_{i=1}^{n} \frac{(X-\mu_i)^2}{\sigma_i^2}.$$
Show that χ_n^2, will be distributed as $G(1/2, n/2)$, which have a p.d.f.
$$f(\chi_n^2) = \frac{1}{2^{n/2}\Gamma(n/2)} (\chi_n^2)^{n/2-1} e^{(-\frac{1}{2}\chi_n^2)}.$$
This is known as the *chi-squared distribution* of order n and has numerous applications in statistics.

2. Let $X_1, X_2, \cdots, X_{10}$ be independent Poisson random variables with mean 1. Use the central limit theorem to approximate $P((X_1 + X_2 + \cdots + X_{10}) \geq 15)$.

3. Let X_i, $i = 1, 2, \cdots, 10$ be independent random variables, each being uniformly distributed over $(0, 1)$. Estimate $P((X_1 + X_2 + \cdots + X_{10}) > 7)$.
Answer: 0.0139

4. The lifetime of a special type of battery is a random variable with mean 40 hours and standard deviation 20 hours. A battery is used until it fails, at which point it is replaced by a new one. Assuming a stockpile of 25 such batteries, the lifetimes of which are independent, approximate the probability that over 1100 hours of use can be obtained.
Answer: 0.1587

5. Let X be the number of times that a fair coin, flipped 40 times, lands heads. Find the probability that $X = 20$. Use the normal approximation and then compare it to the exact solution.
Answer: 0.1272

6. The ideal size of a first-year class at a particular college is 150 students. The college, knowing from past experience that, on the average, only 30 percent of those accepted for admission will actually attend, uses a policy of approving the applications of 450 students. Compute the probability that more than 150 first-year students attend this college.
Answer: 0.06

7. A College would like to have 1050 freshmen. This college cannot accommodate more than 1060. Assume that each applicant accepts with probability 0.6 and that the acceptances can be modeled by Bernoulli trials. If the college accepts 1700, what is the probability that it will have too many acceptances?
Answer: 0.0216

8. A die is rolled 420 times. What is the probability that the sum of the rolls lies between 1400 and 1550?
Answer: 0.967

9. We load on a plane 100 packages whose weights are independent random variables that are uniformly distributed between 5 and 50 pounds. What is the probability that the total weight will exceed 3000 pounds?
Answer: 0.0274

10. The National Health and Nutrition Examination Survey of 19881994 (NHANES III, A-1) estimated the mean serum cholesterol level for U.S. females aged 2074 years to be 204 mg/dl. The estimate of the standard deviation was approximately 44. Using these estimates as the mean m and standard deviation s for the U.S. population, consider the sampling distribution of the sample mean based on samples of size 50 drawn from women in this age group. What is the mean of the sampling distribution? The standard error?
Answer: 204 and 6.2225

11. Suppose the 45 percent of the population favors a certain candidate in an upcoming election. If a random sample size 200 is chosen, find
   (a) the expected value and standard deviation of the number of members of the sample that favor the candidate
   (b) the probability that more than half the members of the sample favor the candidate.

   Answer: 0.0678

12. Suppose cars arrive at a parking lot at a rate of 50 per hour. Lets assume that the process is a Poisson random variable with $\lambda = 50$. Compute the probability that in the next hour the number of cars that arrive at this parking lot will be between 54 and 62.
Answer: 0.2718

13. An insurance company has $25,000$ automobile policy holders. If the yearly claim of a policy holder is a random variable with mean 320 and standard deviation 540, approximate the probability that the total yearly claim exceeds 8.3 million.
Answer: 0.00023.

14. In a Municipal election to select the post of the Chairman, suppose 50% of the population supports Arjun, 20% supports Sarika, and the rest are split between Nisar, John and Rohini. A poll asks 400 random people who they support. Use the central limit theorem to estimate the probability that less than 25% of those polled prefer Nisar, John and Rohini?
Answer: 0.0145.

15. If the mean and standard deviation of serum iron values for healthy men are 120 and 15 micrograms per 100 ml, respectively, what is the probability that a random sample of 50 normal men will yield a mean between 115 and 125 micrograms per 100 ml?
Answer: 0.9818.