Probability and Statistics
Problem Set III

Notations:

1. $X \sim \text{Bin}(n, p)$ indicates that the random variable X has Binomial distribution with n trials and success probability p.

2. $X \sim \text{NB}(r, p)$ indicates that the random variable X has negative Binomial distribution with r successes and success probability p.

3. $X \sim \text{Ge}(p)$ indicates that the random variable X has Geometric distribution with success probability p.

4. $X \sim \text{Hyp}(a, n, N)$ indicates that the random variable X has Hyper geometric distribution, where N is the total number of items, a is the number of success and n is the number of selected items .

5. $X \sim \text{P}(\lambda)$ indicates that the random variable X has Poisson distribution with parameter $\lambda > 0$.

6. $X \sim U(\{x_1, x_2, x_3, \cdots, x_N\})$ indicates that the random variable X has Uniform distribution on the set $\{x_1, x_2, x_3, \cdots, x_N\}$.

7. $X \sim U(\alpha, \beta)$ indicates that the random variable X has Uniform distribution on the interval (α, β).

8. $X \sim G(\alpha, \lambda)$ indicates that the random variable X has Gamma distribution with parameters $\alpha > 0$ & $\lambda > 0$.

9. $X \sim \text{Exp}(\lambda)$ indicates that the random variable X has Exponential distribution with parameters $\lambda > 0$.

10. $X \sim N(\mu, \sigma^2)$ indicates that the random variable X has Normal (Gaussian) distribution with parameters $\mu \in \mathbb{R}$ & $\sigma > 0$.

1. Let X be a random variable. Define $X_{(m)} = X(X-1)(X-2)\cdots(X-m+1)$, for $m \in \mathbb{N}$. Then find the expectation $E(X_{(m)})$ of $X_{(m)}$ for the following cases:

   (a) $X \sim \mathrm{Bin}(n,p)$. **Answer:** $= n(n-1)(n-2)\cdots(n-m+1)p^m$

   (b) $X \sim \mathrm{NB}(r,p)$. **Answer:** $= r(r+1)(r+2)\cdots(r+m-1)(\frac{1-p}{p})^m$

   (c) $X \sim \mathrm{Hyp}(a,n,N)$. **Answer:** $= \dfrac{\binom{N-m}{n-m}}{\binom{N}{n}} a(a-1)(a-2)\cdots(a-m+1)$

   (d) $X \sim P(\lambda)$. **Answer:** $= \lambda^m$

2. A person has to open a lock whose key is lost among a set of N keys. Assume that out of these N keys only one can open the lock. To open the lock the person tries keys one by one by choosing, at each attempt, one of the keys at random from the unattempted keys. The unsuccessful keys are not considered for future attempts. Let Y denote the number of attempts the person will have to make to open the lock. Show that $Y \sim U(\{1,2,3,\cdots,N\})$ and hence find the mean and the variance of the r.v. Y. **Answer:** $E(Y) = \frac{N+1}{2}$ and $Var(Y) = \frac{N^2-1}{12}$.

3. Each child in a family is equally likely to be a boy or a girl. Find the minimum number of children the family should have so that the probability of it having at least a boy and at least a girl is at least 0.95. (Use Binomial distribution)
   Answer: 6

4. There are 30 applicants for a job, out of which only 20 applicants are qualified for the job. Six applicants are selected at random from these 30 applicants. Find the probability that, among the selected candidates, at least two will be qualified for the job. (Use Hyper geometric distribution)
   Answer: $1 - \dfrac{\binom{20}{0}\binom{10}{6}}{\binom{30}{6}} - \dfrac{\binom{20}{1}\binom{10}{5}}{\binom{30}{6}}$

5. The probability of hitting a target in each shot is 0.002. Find the approximate probability of hitting a target at least twice in 2000 shots. (Use Poisson distribution)
   Answer: $1 - 5e^{-4}$

6. Eighteen balls are placed at random in seven boxes that are labeled $B_1, \cdots, B_7$. Find the probability that boxes with labels B_1, B_2 and B_3 all together contain six balls.
   Answer: $\binom{18}{6}(\frac{3}{7})^6(\frac{4}{7})^{12}$

7. (i) Suppose a group of 100 men aged $60-64$ in Dehradun received a new flu vaccine from a health center in 2014. From the 2014 life table of the health center, it is found that the approximate probability that a man, aged between $60-64$, dies in the next year is 0.02. How likely are, at least 5 out of 100 men who received flu vaccine and aged $60-64$ to die within the next year?
   (ii) What is the probability that amongst the 60 to 64-year old men who got flu

vaccination exactly 25 survive and at least 10 die within the next year? (*you don't need to calculate the exact numerical values of the probabilities*)
Answer: $(i)\, 1-\sum_{i=0}^{4}\binom{100}{i}(.02)^i(0.98)^{100-i}$, $\quad (ii)\, \binom{100}{25}(.98)^{25}(.02)^{75}$.

8. Two teams (say Team A and Team B) play a series of games until one team wins 5 games. If the probability of Team A (Team B) winning any game is 0.7 (0.3), find the probability that the series will end in 8 games.
Answer: 0.188 (approximately).

9. Let $X \sim \mathrm{Ge}(p)$. Find $E(\frac{1}{2^X})$.
Answer: $\frac{p}{p+1}$

10. The *characteristic function* (CF) of a random variable X is defined as

$$C_X(t) = E[e^{itX}] = \begin{cases} \sum_j e^{itx_j} p(x_j) & \text{for discrete case} \\ \int e^{itx} f(x)dx & \text{for continuous case} \end{cases}$$

where $i = \sqrt{-1}$. Show that the characteristic function for Binomial distribution $Bin(n,p)$ and Poisson distribution with mean λ are (i) $(pe^{it} + (1-p))^n$ and (ii) $e^{\lambda(e^{it}-1)}$.

11. The *cumulunt generating function* is defined as $K_X(t) = ln(\phi_X(t))$, where ϕ_X is the moment generating function of X. Show that $K'(0) = E[X]$, $K''(0) = Var(X)$.

12. Let $X \sim U(0,k)$, where k is a positive integer and $Y = X - [X]$, where $[X]$ is the largest integer $\leq X$. Show that $Y \sim U(0,1)$.

13. Let $X \sim U(0,1)$, where k is a positive integer and $Y = -\ln(1-X)$. Find the c.d.f. and p.d.f. of Y.

14. Alvin's driving time to work is between 15 and 20 minutes if the day is sunny, and between 20 and 25 minutes if the day is rainy, with all times being equally likely in each case. Assume that a day is sunny with probability 2/3 and rainy with probability 1/3. What is the p.d.f. of the driving time, viewed as a random variable X?

15. If X is described by a Gaussian distribution of mean μ and variance σ^2, calculate the probabilities that X lies within 1σ, 2σ and 3σ of the mean.
Answer: $0.6826, 0.9544, 0.9974$
(Hint: Use Normal table (z-table). $\Phi(z=1) = 0.8413$; $\Phi(z=2) = .9772$; $\Phi(z=3) = .9987$).

16. Let $X \sim N(\mu, \sigma^2)$, , for some $\mu \in \mathbb{R}$ and $\sigma > 0$ and $Y = aX + b$, where $a \in \mathbb{R} - \{0\}$ and $b \in \mathbb{R}$. Then show that $Y \sim N(a\mu + b, a^2\sigma^2)$.

17. Let $X \sim N(2,4)$. Find $P(X \leq 0)$, $P(|X| \geq 2)$, $P(1 < X \leq 3)$ and $P(X \leq 3 | X > 1)$.
Answer: 0.1587, 0.5228, 0.383 & 0.5539
(Hint: $F_X(x) = \Phi(\frac{x-\mu}{\sigma})$ and use Normal table (z-table). $\Phi(z=-2) = 0.0228$; $\Phi(z=-1) = 0.1587$; $\Phi(z=0) = 0.5$; $\Phi(z=0.5) = .6915$).

18. Suppose the diameter of a certain car component follows the normal distribution with $X \sim N(10, 3^2)$. Find the proportion of these components that have diameter larger than 13.4 mm ($\Phi(z = 1.13) = 0.8708$).
Answer: 0.1292

19. The number of miles that a particular car can run before its battery wears out is exponentially distributed with an average of $10,000$ miles. The owner of the car needs to take a 5000-mile trip. What is the probability that he will be able to complete the trip without having to replace the car battery?
Answer: $e^{-1/2}$

20. Engineers designing the next generation of space shuttles plan to include two fuel pumps one active, the other in reserve. If the primary pump malfunctions, the second is automatically brought on line. Suppose a typical mission is expected to require that fuel be pumped for at most 50 hours. According to the manufacturer's specifications, pumps are expected to fail once every 100 hours. What are the chances that such a fuel pump system would not remain functioning for the full 50 hours?
Answer: $1 - \frac{3}{2}e^{-1/2}$