# Negative Binomial and Geometric Distribution

## 1. Negative Binomial Distribution

Let $r \in \mathbb{N}$. Suppose that we keep performing independent Bernoulli trials until the $r$-th success is observed. Further suppose that the probability of success in each trial is $p \in (0,1)$. Thus, the sample space is

$\mathcal{S} = \{(w_1, w_2, \ldots, w_n) : n \in \{r, r+1, \ldots\}, w_n = s, w_i \in \{s, f\}, i = 1, 2, \ldots, n-1; r-1$ of $w_1, w_2, \ldots, w_{n-1}$ are $s$ and remaining $n-r$ of $w_1, w_2, \ldots, w_{n-1}$ are $f\}$.

**Note:** To find $r$-th success, we have to perform Bernoulli trials at least $r$-times. Thus, $(w_1, w_2, \ldots, w_n) \in S$ corresponds to one of $\binom{n-1}{r-1}$ ways in which the $r$-th success is obtained in the $n$-th Bernoulli trials $w_n = s$ and the first $n-1$ Bernoulli trials result in $r-1$ successes and $n-r$ failures.

Define the r.v. $X : \mathcal{S} \longrightarrow \mathbb{R}$ by

$$\begin{aligned} X((w_1, w_2, \ldots, w_n)) &= n - r \\ &= \text{number of failures preceding the } r\text{-th success} \end{aligned}$$

Clearly, for $x \notin \{0, 1, 2, \cdots\}$, $P(\{X = x\}) = 0$. Also, for $x \in \{0, 1, 2, \cdots\}$, event $\{X = x\}$ occurs if and only if the $(r+x)$-th trial results in success and the first $(r+x-1)$ Bernoulli trials result in $r-1$ successes and $x$ failures are observed. Since the trials are independent, for $x \in \{0, 1, 2, \cdots\}$, we have

$$P(\{X = x\}) = p_1 p_2,$$

where $p_1$ is the probability of observing $(r-1)$ successes in the first $(r+x-1)$ independent Bernoulli trials and $p_2$ is the probability of getting the success on the $(r+x)$-th trial. Clearly, $p_2 = p$, and using the property of Binomial distribution

$$p_1 = \binom{r+x-1}{r-1} p^{r-1} (1-p)^x.$$

Therefore, $x \in \{0, 1, 2, \cdots\}$,

$$P(\{X = x\}) = \binom{r+x-1}{r-1} p^r (1-p)^x.$$

Thus, the r.v. $X$ is of discrete type with support $E_X = \{0, 1, 2, \cdots\}$ and p.m.f.

$$f_X(x) = P(\{X = x\}) = \begin{cases} \binom{r+x-1}{r-1} p^r (1-p)^x, & \text{if } x \in \{0, 1, 2, \cdots\} \\ 0, & \text{otherwise} \end{cases}. \tag{1}$$

The random variable $X$ is called a Negative Binomial random variable with $r$ successes and success probability $p \in (0,1)$ and it is written as $X \sim \mathrm{NB}(r, p)$. The probability distribution with the p.m.f. (1) is called a Negative Binomial distribution with $r$ successes and success probability $p \in (0,1)$.

**Remark 1.** (1) *It is easy to see that the series* $\sum_{x=0}^{\infty} \binom{r+x-1}{r-1} t^x$ *is an absolutely convergent series, for* $|t| < 1$ *and* $\sum_{x=0}^{\infty} \binom{r+x-1}{r-1} t^x = (1-t)^{-r}$, *for* $|t| < 1$. *Thus,* $\sum_{x \in E_X} f_X(x) = p^r \sum_{x=0}^{\infty} \binom{r+x-1}{r-1} (1-p)^x = p^r (1-(1-p))^{-r} = 1$.

(2) *Consider a sequence of independent Bernoulli trials with probability of success in each trial being* $p$. *Let* $Z$ *denote the number of trials required to get the* $r$*-th success, where* $r \in \mathbb{N}$ *and* $X = Z - r$. *Then* $X \sim NB(r, p)$.

Now, the m.g.f. of $X \sim \mathrm{NB}(r, p)$ is

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \sum_{x \in E_X} e^{tx} f_X(x) \\
&= \sum_{x=0}^{\infty} e^{tx} \binom{r+x-1}{r-1} p^r q^x, \text{ where } q = 1-p \\
&= \sum_{x=0}^{\infty} \binom{r+x-1}{r-1} p^r (qe^t)^x \\
&= p^r (1-qe^t)^{-r}, \ |qe^t| < 1 \\
&= \big(\frac{p}{1-qe^t}\big)^r, \ |t| < -\ln q
\end{aligned}$$

Therefore,

$$\begin{aligned}
&M_X^{(1)}(t) = p^r\{rqe^t(1-qe^t)^{-r-1}\}, \ |t| < -\ln q; \\
&M_X^{(2)}(t) = p^r\{rqe^t(1-qe^t)^{-r-1} + r(r+1)(qe^t)^2(1-qe^t)^{-r-2}\}, \ |t| < -\ln q; \\
&E(X) = M_X^{(1)}(0) = \frac{rq}{p}; \\
&E(X^2) = M_X^{(2)}(0) = \frac{r(r+1)q^2}{p^2} + \frac{rq}{p}; \\
&\text{and } Var(X) = E(X^2) - (E(X))^2 = \frac{rq}{p^2}, \text{ where } q = 1-p.
\end{aligned}$$

**Example 2.** *A person repeatedly rolls a fair dice independently until an upper face with two or three dots is observed twice. Find the probability that the person would require eights rolls to achieve this.*

**Solution:** In each trial, let us label the outcome of observing an upper face with two or three dots as success and observing any other outcome as a failure. Hence success probability in each trial is $\frac{1}{3}$. Let $Z$ denote the number of trials required to get the second success and $X = Z - 2$. Then $X \sim \mathrm{NB}(2, \frac{1}{3})$. Therefore, the required probability is

$$P(\{Z = 8\}) = P(\{X = 6\}) = \binom{7}{1}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^6 = \frac{448}{6561}.$$

**Example 3.** *A mathematician carries one matchbox each in his right and left pockets. When he wants a match, he selects the left pocket with probability $p$ and the right pocket with probability $1-p$. Suppose that initially each box contains $N$ matches. Consider the moment when the mathematician for the first time discovers that one of the match boxes is empty. Find the probability that at that moment the other box contains exactly $k$ matches, where $k \in \{0, 1, 2, \cdots, N\}$.*

**Solution:** Let us identify success with the choice of the left pocket. The left pocket box will be empty at the moment when the right pocket box contains exactly $k$ matches if and only if $N-k$ failures precede the $(N+1)$-th success. A similar arguments applies to the right pocket.

Now the required probability is

$$\begin{aligned}
p &= P(\text{the left pocket is found empty, the right pocket contains k matches}) \\
&\quad + P(\text{the right pocket is found empty, the left pocket contains k matches}) \\
&= \binom{N+1+N-k-1}{N+1-1} p^{N+1}(1-p)^{N-k} + \binom{N+1+N-k-1}{N+1-1}(1-p)^{N+1}p^{N-k} \\
&= \binom{2N-k}{N} p^{N+1}(1-p)^{N-k} + \binom{2N-k}{N}(1-p)^{N+1}p^{N-k}.
\end{aligned}$$

## 2. Geometric Distribution

An NB$(1,p)$ distribution is called a geometric distribution with success probability $p$ and is denoted by Ge$(p)$. In this case, the sample space is $\mathcal{S} = \{(w_1, w_2, \ldots, w_n) \mid w_n = s, w_1, w_2, \ldots, w_{n-1} \text{ are } f\}$ and the r.v. $X : \mathcal{S} \longrightarrow \mathbb{R}$ is defined as

$$X((w_1, w_2, \ldots, w_n)) = n - 1 = \text{number of failures proceeding the first success.}$$

Hence, the p.m.f. and d.f. of $X$ are

$$f_X(x) = P(\{X = x\}) = \begin{cases} pq^x, & \text{if } x \in \{0, 1, 2, \cdots\} \text{ where } q = 1-p \\ 0, & \text{otherwise} \end{cases},.$$

and

$$\begin{aligned}
F_X(x) &= P(\{X \leq x\}) \\
&= \begin{cases} 0, & \text{if } x < 0 \\ p\sum_{x=0}^{k} q^x, & \text{if } k \leq x < k+1, \text{ where } k = 0, 1, \cdots \end{cases} \\
&= \begin{cases} 0, & \text{if } x < 0 \\ 1 - q^{k+1}, & \text{if } k \leq x < k+1, \text{ where } k = 0, 1, \cdots \end{cases}.
\end{aligned}$$

respectively. Also, $M_X(t) = \frac{p}{1-qe^t}$, $|t| < -\ln q$, $E(X) = \frac{q}{p}$ and $Var(X) = \frac{q}{p^2}$.

**Remark 4.** *Suppose the r.v. $X : \mathcal{S} \longrightarrow \mathbb{R}$ is given by*

$$X((w_1, w_2, \ldots, w_n)) = n = \textit{number of trials to get the first success.}$$

*Hence, the p.m.f. of $X$ is*

$$f_X(x) = P(\{X = x\}) = \begin{cases} pq^{x-1}, & \textit{if } x \in \{1, 2, \cdots\} \textit{ where } q = 1-p \\ 0, & \textit{otherwise} \end{cases},.$$

*So, in this case the m.g.f. of $X$ is*

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \sum_{x \in E_X} e^{tx} f_X(x) \\
&= \sum_{x=1}^{\infty} e^{tx} pq^{x-1}, \\
&= pe^t \sum_{x=0}^{\infty} (qe^t)^x \\
&= \frac{pe^t}{1-qe^t}, \quad |t| < -\ln q
\end{aligned}$$

*Therefore,*

$$M_X^{(1)}(t) = \frac{pe^t}{(1-qe^t)^2},\ |t| < -\ln q;$$
$$M_X^{(2)}(t) = \frac{(1-qe^t)^2pe^t - 2pe^t(1-qe^t)(-qe^t)}{(1-qe^t)^4},\ |t| < -\ln q;$$
$$E(X) = M_X^{(1)}(0) = \frac{1}{p};$$
$$E(X^2) = M_X^{(2)}(0) = \frac{1+q}{p^2};$$
$$\textit{and } Var(X) = E(X^2) - (E(X))^2 = \frac{q}{p^2},\ \textit{where } q = 1-p.$$