Moment generating function and Moment Inequalities

Let X be a random variable and let $A = \{t \in \mathbb{R} \mid E(e^{tX}) \text{ is finite}\}$. The function $M_X : A \longrightarrow \mathbb{R}$, defined by

$$M_X(t) = E(e^{tX})$$

is known as the moment generating function (m.g.f.) of the random variable X if $E(e^{tX})$ is finite on an interval $(-a, a) \subseteq A$, for some $a > 0$.

Theorem 1. *Let X be a random variable with the moment generating function (m.g.f.) M_X that is finite on an interval $(-a, a)$, for some $a > 0$. Then*

(1) *for each $r \in \{1, 2, \cdots\}$, $M_X^{(r)}(t)$ exists on $(-a, a)$, and for each $r \in \{1, 2, \cdots\}$, $\mu'_r = E(X^r)$ is finite and is equal to $\mu'_r = E(X^r) = M_X^{(r)}(0)$, where $M_X^{(r)}(t) = \frac{d^r M_X(t)}{dt^r}$;*

(2) $M_X(t) = \sum\limits_{r=0}^{\infty} \frac{t^r}{r!}\mu'_r$, *$t \in (-a, a)$.*

Example 2. *Let X be a random variable with the p.m.f.*

$$f_X(k) = \begin{cases} \frac{6}{\pi^2 k^2}, & \text{if } k \in \{1, 2, \cdots\} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\frac{1}{\pi^2} \sum\limits_{k=1}^{\infty} \frac{e^{tk}}{k^2}$ is not convergent for every $t > 0$. Thus the moment generating function (m.g.f.) of the random variable X does not exist.

Example 3. *Let X be a random variable with p.d.f.*

$$f_X(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Now, the moment generating function (m.g.f.) of the random variable X

$$M_X(t) = E(e^{tX}) = \frac{1}{2} \int\limits_0^{\infty} e^{(t-1/2)x} dx = \frac{1}{1-2t}, \ t < \frac{1}{2}.$$

Also $M_X^{(1)}(t) = \frac{2}{(1-2t)^2}$ and $M_X^{(2)}(t) = \frac{8}{(1-2t)^3}$, $t < \frac{1}{2}$. It follows that

$$E(X) = 2, E(X^2) = 8, \text{ and } Var(X) = 4$$

Proposition 4. *Let X be a continuous random variable that takes only non-negative values with a p.d.f. f_X. Then there exists a p.d.f. g_X of X such that $g_X(x) = 0$, for $x < 0$.*

Proof. Since X takes only non-negative values, $P(\{X < 0\}) = 0$. Then the function $g_X : \mathbb{R} \longrightarrow \mathbb{R}$, defined by,

$$g_X(x) = \begin{cases} f_X(x), & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

is a p.d.f. of X (verify!). □

Theorem 5. ***(Markov's Inequality)*** *If X is random variable that takes only non-negative values, then for any $a > 0$,*

$$P(\{X \geq a\}) \leq \frac{E(X)}{a}.$$

Proof. Suppose X is a continuous random variable with a p.d.f. f of X such that $f(x) = 0$, for $x < 0$. Then

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{0}^{\infty} xf(x)dx \\ &= \int_{0}^{a} xf(x)dx + \int_{a}^{\infty} xf(x)dx \\ \Rightarrow E(X) &\geq \int_{a}^{\infty} xf(x)dx \\ \Rightarrow E(X) &\geq a\int_{a}^{\infty} f(x)dx \\ \Rightarrow E(X) &\geq aP(X \geq a) \\ \Rightarrow P(\{X \geq a\}) &\leq \frac{E(X)}{a}. \end{aligned}$$

□

General form of Markov Inequality: Suppose that $E(|X|^r) < \infty$, for some $r > 0$. Then, for any any $a > 0$,

$$P(\{|X| \geq a\}) \leq \frac{E(|X|^r)}{a^r}.$$

Corollary 6. ***(Chebyshev Inequality)*** *Suppose that random variable has finite first two moments. If $\mu = E(X)$ and $\sigma^2 = Var(X)$. Then, for any any $a > 0$,*

$$P(\{|X - \mu| \geq a\}) \leq \frac{\sigma^2}{a^2}.$$

Proof. We have $P(\{|X-\mu| \geq a\}) = P(\{|X-\mu|^2 \geq a^2\})$ (verify it). Using the Markov Inequality on the random variable $|X - \mu|^2$, we have

$$\begin{aligned} P(\{|X - \mu|^2 \geq a^2\}) &\leq \frac{E((X - \mu)^2)}{a^r} = \frac{\sigma^2}{a^r} \\ \Rightarrow P(\{|X - \mu| \geq a\}) &\leq \frac{\sigma^2}{a^r}. \end{aligned}$$

□

Example 7. *Let X be a random variable with the p.m.f.*

$$f_X(x) = \begin{cases} \frac{1}{8}, & \text{if } x \in \{-1, 1\} \\ \frac{1}{4}, & \text{if } x = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then $E(X) = \sum\limits_{x \in S_X} xf_X(x) = 0$ and $E(X^2) = \sum\limits_{x \in S_X} x^2 f_X(x) = \frac{1}{4}$. Therefore, using the Markov Inequality, we have

$$P(\{|X| \geq 1\}) \leq \frac{E(X^2)}{1} = \frac{1}{4}.$$

The exact probability is

$$P(\{|X| \geq 1\}) = P(\{X \in \{-1, 1\}\}) = \frac{1}{4}.$$

Definition 8. *A random variable X is said to have a symmetric distribution about a point $\mu \in \mathbb{R}$ if $P(\{X \leq \mu+x\}) = P(\{X \geq \mu-x\})$, $\forall\, x \in \mathbb{R}$, i.e., $F_X(\mu-x)+F_X(\mu+x) = 1$, $\forall\, x \in \mathbb{R}$.*

Remark 9. *Let X be a random variable having p.d.f./p.m.f. f_X and $\mu \in \mathbb{R}$. Then the distribution of X is symmetric about μ if and only if $f_X(\mu - x) = f_X(\mu + x)$, $\forall\, x \in \mathbb{R}$.*