Function of Random Variables

Let $(\mathcal{S}, \Sigma, P)$ be a probability space with a random variable $X : \mathcal{S} \longrightarrow \mathbb{R}$, and let $h : \mathbb{R} \to \mathbb{R}$ be a function. Consider the function $Z : \mathcal{S} \to \mathbb{R}$ defined by $Z(w) = h(X(w))$. It will be interesting to know when Z is an RV and what are the probabilistic properties of Z.

Remark 1.
(1) *Let X be a random variable and let $h : \mathbb{R} \to \mathbb{R}$ be a function such that $h^{-1}(A) \in \mathbb{B}_{\mathbb{R}}$ for every $A \in \mathbb{B}_{\mathbb{R}}$. Then the function $Z : \mathcal{S} \to \mathbb{R}$ defined by $Z(w) = h(X(w))$ is a random variable. Prove! The function Z (written as $h \circ X$ or $h(X)$) is called a function of random variable X.*
(2) *If X is an RV, and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $h \circ X$ is an RV. In particular, X^2, $|X|$, $\max\{X, 0\}$ and $\sin X$ are random variables.*
(3) *If X is an RV and h is strictly monotone then $h(X)$ is an RV.*

The next two theorems deal with probability distribution of a function of random variables.

Theorem 2. *Let X be an RV of discrete type with support E_X and p.m.f. f_X. Let $h : \mathbb{R} \to \mathbb{R}$ be a function such that $h^{-1}(A) \in \mathbb{B}_{\mathbb{R}}$ for every $A \in \mathbb{B}_{\mathbb{R}}$ and let $Z : \mathcal{S} \to \mathbb{R}$ be a function defined by $Z(w) = h(X(w))$. Then Z is an RV of discrete type with support $E_Z = \{h(x) : x \in E_X\}$ and p.m.f.*

$$f_Z(z) = \begin{cases} \sum_{x \in A_z} f_X(x), \text{ if } z \in E_Z, \\ 0, \text{ otherwise} \end{cases}$$
$$= \begin{cases} P(\{X \in A_z\}), \text{ if } z \in E_Z, \\ 0, \text{ otherwise,} \end{cases}$$

where $A_z = \{x \in E_X : h(x) = z\}$.

Problem 3. *Let X be a random variable with p.m.f.*

$$f_X(x) = \begin{cases} \frac{1}{7}, \text{ if } x \in \{-2, -1, 0, 1\}, \\ \frac{3}{14}, \text{ if } x \in \{2, 3\}, \\ 0, \text{ otherwise.} \end{cases}$$

Show that $Z = X^2$ is a random variable and find its p.m.f.

Solution. Clearly $E_X = \{-2, -1, 0, 1, 2, 3\}$ and $E_Z = \{0, 1, 4, 9\}$. Also,

$$P(\{Z = 0\}) = P(\{X^2 = 0\}) = P(\{X = 0\}) = \tfrac{1}{7},$$

$$P(\{Z = 1\}) = P(\{X^2 = 1\}) = P(\{X \in \{-1, 1\}\}) = \tfrac{1}{7} + \tfrac{1}{7} = \tfrac{2}{7},$$

$$P(\{Z = 4\}) = P(\{X^2 = 4\}) = P(\{X \in \{-2, 2\}\}) = \tfrac{1}{7} + \tfrac{3}{14} = \tfrac{5}{14},$$

$$P(\{Z = 9\}) = P(\{X^2 = 9\}) = P(\{X \in \{-3, 3\}\}) = 0 + \tfrac{3}{14} = \tfrac{3}{14}.$$

Clearly Z is of discrete type.

The p.m.f. of Z is

$$f_Z(z) = \begin{cases} \frac{1}{7}, \text{if } z = 0, \\ \frac{2}{7}, \text{if } z = 1, \\ \frac{5}{14}, \text{if } z = 4, \\ \frac{3}{14}, \text{if } z = 9, \\ 0, \text{otherwise.} \end{cases}$$

Theorem 4. *Let X be a random variable of continuous type with p.d.f. f_X and support E_X such that E_X is a finite union of disjoint open intervals in $\mathbb{R}$. Let $h : \mathbb{R} \to \mathbb{R}$ function such that $h^{-1}(A) \in \mathbb{B}_{\mathbb{R}}$ for every $A \in \mathbb{B}_{\mathbb{R}}$, and h is differentiable and strictly monotone on E_X. Let $E_T = \{h(x) : x \in E_X\}$. Then $T = h(X)$ is an RV of continuous type with p.d.f.*

$$f_T(t) = \begin{cases} f_X(h^{-1}(t)) \left| \frac{d}{dt} h^{-1}(t) \right|, \text{if } t \in E_T, \\ 0, \text{otherwise.} \end{cases}$$

Problem 5. *Let X be an RV with p.d.f.*

$$f_X(x) = \begin{cases} e^{-x} \text{if } x > 0, \\ 0, \text{otherwise.} \end{cases}$$

and let $T = X^2$. Show that T is an RV of continuous type and find its p.d.f. Also, find the p.d.f. of T by computing its c.d.f.

Solution. Clearly T is an RV and $E_X = E_T = (0, \infty)$. Also, $h(x) = x^2$, $x \in E_X$ is strictly increasing on E_X with inverse $h^{-1}(x) = \sqrt{x}$, $x \in E_T$. From the above theorem, T is a continuous type RV with p.d.f.

$$\begin{aligned} f_T(t) &= \begin{cases} f_X(\sqrt{t}) \left| \frac{d}{dt}(\sqrt{t}) \right|, \text{if } t > 0, \\ 0, \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{e^{-\sqrt{t}}}{2\sqrt{t}}, \text{if } t > 0, \\ 0, \text{otherwise.} \end{cases} \end{aligned}$$

Now, let us find the c.d.f. of T.

$$\begin{aligned} F_T(t) &= P(T \leq t) \\ &= P(X^2 \leq t) \\ &= \begin{cases} 0, \text{if } t \leq 0, \\ P(-\sqrt{t} \leq X \leq \sqrt{t}), \text{if } t > 0 \end{cases} \\ &= \begin{cases} 0, \text{if } t \leq 0, \\ \int_0^{\sqrt{t}} f_X(x) dx, \text{if } t > 0 \end{cases} \\ &= \begin{cases} 0, \text{if } t \leq 0, \\ -e^{-\sqrt{t}}, \text{if } t > 0. \end{cases} \end{aligned}$$

We see that F_T is differentiable everywhere except $t = 0$. Therefore, the p.d.f. of T is given by

$$\begin{aligned} f_T(t) &= \begin{cases} (-e^{-\sqrt{t}})', \text{if } t > 0, \\ 0, \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{e^{-\sqrt{t}}}{2\sqrt{t}}, \text{if } t > 0, \\ 0, \text{otherwise.} \end{cases} \end{aligned}$$