Interval Estimation

In the theory of point estimation, we are interested in estimating the value of parametric function $g(\theta)$ by a single value t based on the observations $x_1, \ldots, x_n$ when the samples are drawn from a density $f(x, \theta)$, $\theta \in Theta$. In practice, one is not generally interested in finding a point estimate of $g(\theta)$, but a set of values, say, $H(\theta)$, such that $H(\theta)$ contains the true value of the parameter $g(\theta)$ with high probability. This type of problems are called problems of confidence interval (set) estimation. When $H(\theta)$ is an interval, it is called confidence interval.

Definition 1. *Confidence Interval: Let $\underline{X} = (X_1, \ldots, X_n)$ be a random sample from a population with density function $f(x, \theta)$, $\theta \in \Theta$. Let $T_1 = t_1(X_1, \ldots, X_n)$, $T_2 = t_2(X_1, \ldots, X_n)$ be two statistics satisfying $T1 \le T_2$ such that*

$$P_\theta[T_1 \le g(\theta) \le T_2] = 1 - \alpha \quad \forall \theta \in \Theta \tag{1}$$

where $(1-\alpha)$ does not depend upon θ. Then the random interval (T_1, T_2) is called the $100(1-\alpha)\%$ confidence interval for $g(\theta)$. The quantity $(1-\alpha)$ is called the confidence coefficient of this interval. The statistics T_1, T_2 are respectively called lower and upper confidence limits for $g(\theta)$. For a given sample observation $\underline{x} = (x_1, \ldots, x_n)$, the values of the statistic $T_1(\underline{x})$, $T_2(\underline{x})$ are the confidence limits for $g(\theta)$.

Usually α is taken to be very small quantity, 0.05, 0.01(say) so that $1-\alpha$ is 0.95, 0.99. In some cases, any of the two statistics T_1, T_2 may be a constant; however $T1$, T_2 can not both be constants.

Definition 2. *One-sided confidence interval: Let X be a random sample from a population with pdf $f(x, \theta)$, $\theta \in \Theta$. Let $T_1 = t_1(\underline{X})$ be a statistic such that*

$$P_\theta[T_1 \le g(\theta)] = 1 - \alpha \quad \theta \in \Theta \tag{2}$$

where α does not depend upon θ. Then T_1 is called the on-sided lower confidence limit for $g(\theta)$. Thus, the interval (T_1, ∞) covers $g(\theta)$ with probability $1-\alpha$. Similarly, let $T_2 = t_2(\underline{X})$ be a statistic such that

$$P_\theta[T_2 \ge g(\theta)] = 1 - \alpha \quad \theta \in \Theta \tag{3}$$

where α does not depend on θ. Then T_2 is called the one-sided upper confidence limit for $g(\theta)$. Here, the interval $(-\infty, T_2)$ covers $g(\theta)$ with probability $(1-\alpha)$. Note that θ may be a vector of parameters. In making probability statements like (1), (2) and (3), we do not mean that $g(\theta)$ is a random variable. (1) means that the probability is $(1-\alpha)$ that the random interval (T_1, T_2) will cover $g(\theta)$, where the true value of the parameter θ may be.

Example 3. *Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ population when σ^2 is known. Find a $100(1-\alpha)\%$ confidence interval for μ.*

Solution: It is known that

$$P_\mu\left(\bar{X} - \tau_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + \tau_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha \quad \forall \mu, \tag{4}$$

where, $\tau_{\alpha/2}$ is the upper $100(\alpha/2)\%$ probability point on a standard normal distribution.
Hence the interval $\left(\bar{X} - \tau_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + \tau_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$ is $100(1-\alpha\%)$ confidence interval for μ.

Clearly any random interval $\left(\bar{X} - \tau_{\alpha_1}\frac{\sigma}{\sqrt{n}}, \bar{X} + \tau_{\alpha_2}\frac{\sigma}{\sqrt{n}}\right)$ where $\alpha = \alpha_1 + \alpha_2$ is $100(1-\alpha\%)$ confidence interval for μ. Again

$$P_\mu\left(\bar{X} - \tau_\alpha\frac{\sigma}{\sqrt{n}} \le \mu\right) = 1 - \alpha \quad \forall \mu. \tag{5}$$

Also

$$P_\mu \left(\bar{X} + \tau_\alpha \frac{\sigma}{\sqrt{n}} \geq \mu \right) = 1 - \alpha \quad \forall \mu. \tag{6}$$

Therefore $T_1 = \bar{X} - \tau_\alpha \frac{\sigma}{\sqrt{n}}$, $T_2 = \bar{X} + \tau_\alpha \frac{\sigma}{\sqrt{n}}$, are respectively the lower and upper confidence limits for μ.
In case of discrete random variables, it is evident that it is not possible to construct confidence intervals of exact confidence coefficient $(1 - \alpha)$ for each $0 < \alpha < 1$. In this case, we may construct confidence intervals of confidence coefficient measuring at least $(1 - \alpha)$. The statistics (T_1, T_2) will, therefore, provide confidence limits to a parameter $g(\theta)$ if

$$P_\theta(T_1 \leq g(\theta)) \leq T_2) \geq 1 - \alpha \quad \theta \in \Theta. \tag{7}$$

Similarly, $T_1(T_2)$ will be lower(upper) confidence limit with confidence coefficient $(1-\alpha)$ if $P_\theta(T_1 \leq g(\theta)) \geq 1 - \alpha$ $(P_\theta(T_2 \geq g(\theta)) \geq 1 - \alpha)$ $\forall$ $\theta \in \Theta$.