Functions of several random variables

Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a n-dimensional random vector. Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function such that $g^{-1}(A) \in \mathbb{B}_{\mathbb{R}^n}$, for all $A \in \mathbb{B}_{\mathbb{R}^m}$. Then $Y = g(\underline{X})$ is an m-dimensional random vector.

Suppose joint c.d.f. of $\underline{X} = (X_1, X_2, \cdots, X_n)$ is $F_{\underline{X}}$ and $m = 1$, i.e., $Y = g(\underline{X})$ is a random variable. Let $y \in \mathbb{R}$. Then

$$P(Y \leq y) = P(g(X_1, X_2, \cdots, X_n) \leq y)$$
$$= \begin{cases} \sum\limits_{\{(x_1,x_2,\cdots,x_n):g(X_1,X_2,\cdots,X_n)\leq y\}} P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n), \text{ if } \underline{X} \text{ is of discrete type} \\ \int\limits_{\{(x_1,x_2,\cdots,x_n):g(X_1,X_2,\cdots,X_n)\leq y\}} f(x_1, x_2, \ldots, x_n)dx_1dx_2\cdots dx_n, \text{ if } \underline{X} \text{ is of continuous type} \end{cases}$$

where f is the joint p.d.f. of $\underline{X} = (X_1, X_2, \cdots, X_n)$ in case of continuous type.

Example 1. *Let X_1, X_2 be independent uniform distributions $U(0, 1)$. Find the c.d.f. of $Y = X_1 + X_2$ and hence find the p.d.f. of Y.*

Solution: The joint p.d.f. of (X_1, X_2) is

$$f(x_1, x_2) = \begin{cases} 1, \text{ if } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0, \text{ otherwise} \end{cases}$$

Now, the c.d.f. of Y is

$$F_Y(y) = P(Y \leq y)$$
$$= P(X_1 + X_2 \leq y)$$
$$= \iint\limits_{\{(x_1,x_2):\ x_1+x_2\leq y\}} f(x_1, x_2)dx_1dx_2$$
$$= \begin{cases} 0, \text{ if } y < 0 \\ \int_0^y (\int_0^{y-x_1} dx_2)dx_1, \text{ if } 0 \leq y < 1 \\ 1 - \int_{y-1}^1 (\int_{y-x_2}^1 dx_1)dx_2, \text{ if } 1 \leq y < 2, \text{ (since } P(X_1 + X_2 \leq y) = 1 - P(X_1 + X_2 > y)) \\ 1, \text{ if } y \geq 2 \end{cases}$$
$$= \begin{cases} 0, \text{ if } y < 0 \\ \frac{y^2}{2}, \text{ if } 0 \leq y < 1 \\ \frac{4y-y^2-2}{2}, \text{ if } 1 \leq y < 2 \\ 1, \text{ if } y \geq 2 \end{cases}$$

Hence, p.d.f. of Y is

$$f_Y(y) = \begin{cases} y, \text{ if } 0 \leq y < 1 \\ 2 - y, \text{ if } 1 \leq y < 2 \\ 0, \text{ otherwise} \end{cases}$$

Example 2. *Let $\underline{X} = (X_1, X_2)$ be a continuous random vector with joint p.d.f. is*

$$f(x_1, x_2) = \begin{cases} e^{-x_1}, \text{ if } 0 < x_2 \leq x_1 < \infty \\ 0, \text{ otherwise} \end{cases}$$

. Find p.d.f. of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

Solution: The c.d.f. of Y_1 is

$$\begin{aligned} F_{Y_1}(y) &= P(Y_1 \le y) \\ &= P(X_1 + X_2 \le y) \\ &= \iint\limits_{\{(x_1,x_2):\ x_1+x_2 \le y\}} f(x_1,x_2)dx_1dx_2 \\ &= \begin{cases} 0, \ \text{if } y < 0 \\ \int\limits_0^{\frac{y}{2}} \Big(\int\limits_{x_2}^{y-x_2} dx_1\Big)dx_2, \ \text{if } y \ge 0 \end{cases} \\ &= \begin{cases} 0, \ \text{if } y < 0 \\ (1-e^{-\frac{y}{2}})^2, \ \text{if } y \ge 0 \end{cases} \end{aligned}$$

Hence, p.d.f. of Y_1 is

$$f_Y(y) = \begin{cases} 0, \ \text{if } y < 0 \\ (1-e^{-\frac{y}{2}})e^{-\frac{y}{2}}, \ \text{if } y \ge 0 \end{cases}$$

Now, the c.d.f. of Y_2 is

$$\begin{aligned} F_{Y_2}(y) &= P(Y_2 \le y) \\ &= P(X_1 - X_2 \le y) \\ &= \iint\limits_{\{(x_1,x_2):\ x_1-x_2 \le y\}} f(x_1,x_2)dx_1dx_2 \\ &= \begin{cases} 0, \ \text{if } y < 0 \\ \int\limits_0^{\infty} \Big(\int\limits_{x_2}^{y+x_2} dx_1\Big)dx_2, \ \text{if } y \ge 0 \end{cases} \\ &= \begin{cases} 0, \ \text{if } y < 0 \\ 1-e^{-y}, \ \text{if } y \ge 0 \end{cases} \end{aligned}$$

Hence, p.d.f. of Y_2 is

$$f_Y(y) = \begin{cases} 0, \ \text{if } y < 0 \\ e^{-y}, \ \text{if } y \ge 0 \end{cases}$$

Transformation of Variables Technique:

Theorem 3. *Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a discrete type random vector with support $E_{\underline{X}}$ and the p.m.f. $f_{\underline{X}}$. Let $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function such that $g_i^{-1}(A) \in \mathbb{B}_{\mathbb{R}^n}$, for all $A \in \mathbb{B}_{\mathbb{R}}$ and $Y_i = g_i(\underline{X})$, $i = 1, 2, \cdots, k$. Define, for $\underline{y} = (y_1, y_2, \cdots, y_k) \in \mathbb{R}^k$, $A_{\underline{y}} = \{\underline{x} = (x_1, x_2, \cdots, x_n) \in E_{\underline{X}} \mid g_1(\underline{x}) \le y_1, \ldots, g_k(\underline{x}) \le y_k\}$ and $B_{\underline{y}} = \{\underline{x} = (x_1, x_2, \cdots, x_n) \in E_{\underline{X}} \mid g_1(\underline{x}) = y_1, \ldots, g_k(\underline{x}) = y_k\}$. Then the random vector $\underline{Y} = (Y_1, Y_2, \cdots, Y_k)$ is of discrete type with joint c.d.f.*

$$F_{\underline{Y}}(\underline{y}) = \sum_{\underline{x} \in A_{\underline{y}}} f_{\underline{X}}(\underline{x}), \ \underline{y} \in \mathbb{R}^k$$

and the p.m.f.

$$f_{\underline{Y}}(\underline{y}) = \sum_{\underline{x} \in B_{\underline{y}}} f_{\underline{X}}(\underline{x}), \ \underline{y} \in \mathbb{R}^k.$$

Example 4. *Let X_1, X_2 be independent random variables with $X_1 \sim Bin(n_1, \theta)$ and $X_2 \sim Bin(n_2, \theta)$, where $n_1, n_2 \in \mathbb{N}$. Without using the m.g.f. of $Y = X_1 + X_2$, find the p.m.f. of Y.*

Solution: The joint p.m.f. of $\underline{X} = (X_1, X_2)$ is given by

$$
\begin{aligned}
f_{\underline{X}}(x_1, x_2) &= f_{X_1}(x_1) f_{X_2}(x_2) \\
&= \begin{cases} \prod_{i=1}^{2} \binom{n_i}{x_i} \theta^{x_i} (1-\theta)^{n_i - x_i}, & \text{if } (x_1, x_2) \in \prod_{i=1}^{2} \{0, 1, \cdots, n_i\} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \left(\prod_{i=1}^{2} \binom{n_i}{x_i} \right) \theta^{x_1 + x_2} (1-\theta)^{(n_1+n_2)-(x_1+x_2)}, & \text{if } (x_1, x_2) \in \prod_{i=1}^{2} \{0, 1, \cdots, n_i\} \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

where $\prod_{i=1}^{2} \{0, 1, \cdots, n_i\} = \{0, 1, \cdots, n_1\} \times \{0, 1, \cdots, n_2\}$. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined by $g(x_1, x_2) = x_1 + x_2$. Then $Y = g(\underline{X}) = X_1 + X_2$. Thus, for $y \notin \{0, 1, \cdots, n_1 + n_2\}$, $B_y = \{(x_1, x_2) \in E_{\underline{X}} \mid x_1 + x_2 = y\} = \emptyset$ and $f_Y(y) = P(Y = y) = 0$. Now, for $y \in \{0, 1, \cdots, n_1 + n_2\}$, we have

$$
\begin{aligned}
f_Y(y) = P(Y = y) &= \sum_{\underline{x} \in B_y} f_{\underline{X}}(x_1, x_2) \\
&= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2}{}_{x_1+x_2=y} \left(\prod_{i=1}^{2} \binom{n_i}{x_i} \right) \theta^{x_1+x_2} (1-\theta)^{(n_1+n_2)-(x_1+x_2)} \\
&= \sum_{x_1=0}^{y} \binom{n_1}{x_1} \binom{n_2}{y - x_1} \theta^y (1-\theta)^{(n_1+n_2)-y} \\
&= \binom{n_1 + n_2}{y} \theta^y (1-\theta)^{(n_1+n_2)-y}
\end{aligned}
$$

Therefore, the p.m.f. of Y is

$$
f_Y(y) = \begin{cases} \binom{n}{y} \theta^y (1-\theta)^{n-y}, \text{if } y \in \{0, 1, \cdots, n\} \\ 0, \text{ otherwise} \end{cases}
$$

where $n = n_1 + n_2$.

Theorem 5. *Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be an $n-$dimensional random vector of continuous type with joint p.d.f. $f_{\underline{X}}$.*

(1) *Let*

$$
\begin{aligned}
y_1 &= g_1(x_1, x_2, \ldots, x_n) \\
y_2 &= g_2(x_1, x_2, \ldots, x_n) \\
&\vdots \\
y_n &= g_n(x_1, x_2, \ldots, x_n)
\end{aligned}
$$

be a one-to-one mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$ sending $(x_1, x_2, \ldots, x_n)$ to $(y_1, y_2, \ldots, y_n)$. That is, there exists the inverse transformation

$$
\begin{aligned}
x_1 &= h_1(y_1, y_2, \ldots, y_n) \\
x_2 &= h_2(y_1, y_2, \ldots, y_n) \\
&\vdots \\
x_n &= h_n(y_1, y_2, \ldots, y_n)
\end{aligned}
$$

defined over the range of the transformation.

In other words, the equations $y_i = g_i(x_1, x_2, \ldots, x_n)$, $1 \le i \le n$ have a unique solution $x_i = h_i(y_1, y_2, \ldots, y_n)$, $1 \le i \le n$.

(2) *Assume that both the mapping and its inverse are continuous.*

(3) *Assume that the partial derivatives*

$$\frac{\partial x_i}{\partial y_j}, 1 \le i \le n, 1 \le j \le n$$

exist and are continuous.

(4) *Assume that the Jacobian J of the inverse transformation*

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} \ne 0$$

for $(y_1, y_2, \ldots, y_n)$ in the range of the transformation.

Then $\underline{Y} = (Y_1, Y_2, \cdots, Y_n) = (g_1(X_1, X_2, \cdots, X_n), g_2(X_1, X_2, \cdots, X_n), \cdots, g_n(X_1, X_2, \cdots, X_n))$ *is a continuous random vector with joint p.d.f.*

$$f_{\underline{Y}}(y_1, y_2, \cdots, y_n) = f_{\underline{X}}(h_1(y_1, y_2, \cdots, y_n), h_2(y_1, y_2, \cdots, y_n), \cdots, h_n(y_1, y_2, \cdots, y_n))|J|$$

Example 6. *Let X_1 and X_2 be independent uniform distributions $U(0,1)$. Find p.d.f. of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.*

Solution: The joint p.d.f. of (X_1, X_2) is given by

$$f(x_1, x_2) = \begin{cases} 1, & \text{if } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$. Then $x_1 = \frac{y_1+y_2}{2}$ and $x_2 = \frac{y_1-y_2}{2}$ and

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = -\frac{1}{2}$$

Therefore the joint p.d.f. of $\underline{Y} = (Y_1, Y_2)$ is

$$f_{\underline{Y}}(y_1, y_2) = f(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2})|J| = \begin{cases} \frac{1}{2}, & \text{if } 0 < y_1 + y_2 < 2, 0 < y_1 - y_2 < 2 \\ 0, & \text{otherwise} \end{cases}$$

Now, the marginal p.d.f. of Y_1 is

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f_{\underline{Y}}(y_1, y_2) dy_2 \\ &= \begin{cases} \int_{-y_1}^{y_1} \frac{1}{2} dy_2, & \text{if } 0 < y \le 1 \\ \int_{y_1-2}^{2-y_1} \frac{1}{2} dy_2, & \text{if } 1 < y < 2 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} y_1, & \text{if } 0 < y \le 1 \\ 2 - y_1, & \text{if } 1 < y < 2 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

The marginal p.d.f. of Y_2 is

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{\underline{Y}}(y_1, y_2) dy_1$$

$$= \begin{cases} \int_{-y_2}^{2+y_2} \frac{1}{2} dy_2, \text{ if } -1 < y_2 \leq 0 \\ \int_{y_2}^{2-y_2} \frac{1}{2} dy_2, \text{ if } 0 < y_2 < 1 \\ 0, \text{ otherwise} \end{cases}$$

$$= \begin{cases} 1 + y_2, \text{ if } -1 < y_2 \leq 0 \\ 1 - y_2, \text{ if } 0 < y_2 < 1 \\ 0, \text{ otherwise} \end{cases}$$

Example 7. *Let X_1 and X_2 be independent exponential distributions Exp(λ). Show that $\frac{X_1}{X_1+X_2} \sim U(0,1)$.*

Solution: The joint p.d.f. of (X_1, X_2) is given by

$$f(x_1, x_2) = \begin{cases} \lambda^2 e^{-\lambda(x_1+x_2)}, \text{ if } x_1 > 0, x_2 > 0 \\ 0, \text{ otherwise} \end{cases}$$

Let $y_1 = \frac{x_1}{x_1+x_2}$ and $y_2 = x_1 + x_2$. Then $x_1 = y_1 y_2$ and $x_2 = (1 - y_1) y_2$ and

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = y_2 \neq 0 \text{ as } x_1 > 0 \text{ and } x_2 > 0$$

Therefore the joint p.d.f. of $\underline{Y} = (Y_1, Y_2)$ is

$$f_{\underline{Y}}(y_1, y_2) = f(y_1 y_2, (1 - y_1) y_2) |J|$$

$$= \begin{cases} \lambda^2 y_2 e^{-\lambda y_2}, \text{ if } y_1 y_2 > 0, (1 - y_1) y_2 > 0 \\ 0, \text{ otherwise} \end{cases}$$

$$= \begin{cases} \lambda^2 y_2 e^{-\lambda y_2}, \text{ if } 0 < y_1 < 1, y_2 > 0 \\ 0, \text{ otherwise} \end{cases}$$

The marginal p.d.f. of Y_1 is

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{\underline{Y}}(y_1, y_2) dy_2$$

$$= \begin{cases} \int_0^{\infty} \lambda^2 y_2 e^{-\lambda y_2} dy_2, \text{ if } 0 < y_1 < 1 \\ 0, \text{ otherwise} \end{cases}$$

$$= \begin{cases} 1, \text{ if } 0 < y_1 < 1 \\ 0, \text{ otherwise} \end{cases}$$

Therefore, $\frac{X_1}{X_1+X_2} \sim U(0,1)$.