Gamma and Exponential Distribution

Consider the improper integral $\int\limits_0^\infty e^{-t}t^{\alpha-1}dt = \int\limits_0^1 e^{-t}t^{\alpha-1}dt + \int\limits_1^\infty e^{-t}t^{\alpha-1}dt$, where $\alpha \in \mathbb{R}$. By Limit comparison test, $\int\limits_0^1 e^{-t}t^{\alpha-1}dt$ converges, for all $\alpha > 0$ and the $\int\limits_1^\infty e^{-t}t^{\alpha-1}dt$ converges, for all $\alpha \in \mathbb{R}$. Hence, the $\int\limits_0^\infty e^{-t}t^{\alpha-1}dt$ is convergent if and only if $\alpha > 0$.

Definition 1. *The function* $\Gamma : (0, \infty) \longrightarrow (0, \infty)$, *defined by,*

$$\Gamma(\alpha) = \int\limits_0^\infty e^{-t}t^{\alpha-1}dt$$

is called the gamma function.

Properties:

(1) $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, $\alpha > 0$.
(2) $\Gamma(n) = (n - 1)!$, $n \in \mathbb{N}$ with the convention that $0! = 1$.
(3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. In general, for $n \in \mathbb{N} \cup \{0\}$, we have $\Gamma(\frac{2n+1}{2}) = \frac{(2n)!}{n!4^n}\sqrt{\pi}$.

1. Gamma Distribution

A continuous random variable X is said to have a gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$ (written as $X \sim G(\alpha, \lambda)$) if probability density function of X is given by

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha e^{-\lambda x} x^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$

Now, the r-th moment of $X \sim G(\alpha, \lambda)$ is

$$\begin{aligned} E(X^r) &= \int\limits_{-\infty}^\infty x^r f_X(x)dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int\limits_0^\infty x^r e^{-\lambda x} x^{\alpha-1}dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int\limits_0^\infty e^{-\lambda x} x^{(\alpha+r)-1}dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^{(\alpha+r)}} \int\limits_0^\infty e^{-t} t^{(\alpha+r)-1}dx, \text{ (by putting } \lambda x = t) \\ &= \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)\lambda^r} \\ &= \frac{\alpha(\alpha + 1)\cdots(\alpha + r - 1)}{\lambda^r} \end{aligned}$$

Hence

$$E(X) = \frac{\alpha}{\lambda};$$
$$E(X^2) = \frac{\alpha(\alpha+1)}{\lambda^2};$$
$$Var(X) = E(X^2) - (E(X))^2 = \frac{\alpha}{\lambda^2}.$$

The m.g.f. of $X \sim G(\alpha, \lambda)$ is

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} e^{-(\lambda-t)x} x^{\alpha-1} dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda-t)^\alpha} \int_0^{\infty} e^{-z} z^{\alpha-1} dz, \text{ if } t < \lambda \text{ (by putting } \lambda - t = z) \\
&= \left(\frac{\lambda}{\lambda-t}\right)^\alpha, \text{ if } t < \lambda.
\end{aligned}$$

Remark 2. *Let $X \sim G(\alpha, \lambda)$ and $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by $h(x) = \lambda x$. Since the support of X is $E_X = (0, \infty)$, the support of $Z = h(X) = \lambda X$ is $E_Z = (0, \infty)$. Clearly, h is strictly increasing on E_X. Therefore, the p.d.f. of $Z = \lambda X$ is*

$$\begin{aligned}
f_Z(z) &= \begin{cases} f_X(h^{-1}(z)) \left|\frac{d}{dz} h^{-1}(z)\right|, & \text{if } z > 0 \\ 0, & \text{if } z \leq 0 \end{cases} \\
&= \begin{cases} \frac{e^{-z} z^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } z > 0 \\ 0, & \text{if } z \leq 0 \end{cases}
\end{aligned}$$

Hence $Z \sim G(\alpha, 1)$.

2. Exponential Distribution

A $G(1, \lambda)$ distribution is called an exponential distribution with parameter $\lambda > 0$ and it is denoted by $\text{Exp}(\lambda)$. Thus p.d.f. of $\text{Exp}(\lambda)$ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

If $X \sim \text{Exp}(\lambda)$, then

$$E(X^r) = \frac{r!)}{\lambda^r};$$
$$E(X) = \frac{1}{\lambda};$$
$$E(X^2) = \frac{2}{\lambda^2};$$
$$Var(X) = E(X^2) - (E(X))^2 = \frac{1}{\lambda^2};$$
$$M_X(t) = \frac{\lambda}{\lambda - t}, \text{ if } t < \lambda.$$

The d.f. of $X \sim \text{Exp}(\lambda)$ is

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)dt \\ &= \begin{cases} \int_0^x \lambda e^{-\lambda t}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases} \\ &= \begin{cases} 1 - e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases} \end{aligned}$$

Remark 3. (1) *A Poisson Process is a model for a series of discrete event where the average time between events is known, but the exact timing of events is random.*
(2) *The exponential distribution occurs naturally if we consider the distribution of the length of intervals between successive events in a Poisson process or, equivalently, the distribution of the interval (i.e. the waiting time) before the first event.*

Example 4. *The waiting time for occurrence of an event E (say repair time of a machine) is exponentially distributed with mean of* 30 *minutes. Find the conditional probability that the waiting time for occurrence of event E is at least* 5 *hours given that it has not occurred in the first* 3 *hours.*

Solution: Let X be the waiting time (in hours) for the occurrence of event E. Then $X \sim \text{Exp}(2)$. Hence, the required probability is $P(\{X > 5\}|\{X > 3\}) = \frac{P(\{X>5\})}{P(\{X>3\})} = \frac{e^{-10}}{e^{-6}} = e^{-4}$.