1.a) mutually exclusive $\Rightarrow P(A \cap B) = 0$ ½

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ①

$0.6 = 0.4 + x$

$x = 0.2$ ½

1.b) independent $\Rightarrow P(A \cap B) = P(A) \cdot P(B)$ ½

$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$ ①

$0.6 = 0.4 + x - 0.4x$

$0.2 = 0.6x$

$x = \frac{1}{3}$ ½

2) D = Person has disease

H = Person is healthy

E = Person test positive

$P(E|H) = 0.05$ $\quad P(E^c|D) = 0.002$

$P(D) = 0.01$, $P(H) = .99$ $\cancel{+.5}$ $\rightarrow$ ②

$$P(D|E) = \frac{P(E|D)\,P(D)}{P(E|D)\,P(D) + P(E|H)\,P(H)}$$ ①

$P(E|D) = 1 - P(E^c|D) = 0.998$ ①

$P(H) = 1 - P(D) = 0.99$

$$P(D|E) = \frac{0.998 \times 0.01}{0.998 \times 0.01 + 0.05 \times 0.99} = \frac{0.0998}{0.5948} = 0.168$$ ①

3.a) $\int_0^1 ax + \int_1^2 a + \int_2^3 -ax + 3a = 1$

$\Rightarrow a\left[\frac{1}{2}\right] + a(2-1) + a\left[-\frac{x^2}{2} + 3x\right]_2^3 = 1$

$\Rightarrow \frac{a}{2} + a + a\left[\frac{1}{2}\right] = 1.$

$\Rightarrow a = \frac{1}{2}$ ①

$$f_X(x) = \begin{cases} \frac{x}{2} & , 0 \le x \le 1 \\ \frac{1}{2} & , 1 \le x \le 2 \\ -\frac{x}{2} + \frac{3}{2} & , 2 \le x \le 3 \\ 0 & , \text{o.w} \end{cases}$$

(b). $$F_X(x) = \begin{cases} \int_0^x \frac{x}{2}\, dx & , 0 \le x \le 1 \\ F(1) + \int_1^x \frac{1}{2}\, dx & , 1 \le x \le 2 \\ F(2) + \int_2^x \left(-\frac{x}{2} + \frac{3}{2}\right) dx & , 2 \le x \le 3 \\ 1 & , x > 3 \end{cases}$$

$$= \begin{cases} \frac{x^2}{4} & 0 \le x \le 1 \\ \frac{x}{2} - \frac{1}{4} & 1 \le x \le 2 \\ -\frac{x^2}{4} + \frac{3}{2}x - \frac{5}{4} & 2 \le x \le 3 \\ 1 & x > 3 \\ 0 & x < 0 \end{cases}$$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

... than 1.5 , $i = 1, 2, 3$

(C) Let X_1, X_2, X_3 be the three independent observations

We need to compute $\rightarrow$ P[exactly 1 of X_1, X_2 & X_3 is greater than 1.5]

Let $A: X_1 > 1.5$, $B: X_2 > 1.5$, $C: X_3 > 1.5$

Hence the required probability is $\rightarrow$

$P[A \cap B^c \cap C^c] \cup [A^c \cap B \cap C^c] \cup [A^c \cap B^c \cap C]]$ —— 1

$= P[A \cap B^c \cap C^c] + P[A^c \cap B \cap C^c] + P[A^c \cap B^c \cap C]$

Now $F_X(1.5) = 1/2 \Rightarrow 1 - F_X(1.5) = 1/2$ —— 1

Also, $P[A \cap B^c \cap C^c] = P[A^c \cap B \cap C^c] = P[A^c \cap B^c \cap C]$.

Since events are independent,

$P(A \cap B^c \cap C^c) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

Hence, the required probability $= 3 \times \frac{1}{8} = \frac{3}{8}$ —— 1

4] Let X : Largest number drawn in 'n' draws with replacement.

$\Rightarrow E_X = \{1, 2, \ldots, N\}$. 1

$$P(X \leq x) = \underbrace{\left(\frac{x}{N}\right) \cdot \left(\frac{x}{N}\right) \cdot \cdots \left(\frac{x}{N}\right)}_{n \text{ times}}$$

[Probability of drawing a ticket less than or equal to 'x' is the same at each draw, and the draws are independent].

$$\Rightarrow P(X \leq x) = \left(\frac{x}{N}\right)^n . \quad 1$$

Hence,

$$F_X(x) = \begin{cases} 0 , & x < 1 \quad 1/2 \\ \left(\frac{\lfloor x \rfloor}{N}\right)^n , & 1 \leq x \leq N \quad 1 \\ 1 , & x > N . \end{cases}$$

The PMF is given by →

$$f_X(x) = P(X = x) = F_X(x) - F_X(x^-) \quad 1/2$$

$$= P(X \leq x) - P(X \leq x-1)$$

$$\Rightarrow f_X(x) = \begin{cases} \left(\frac{x}{N}\right)^n - \left(\frac{x-1}{N}\right)^n , & x \in \{1, 2, \ldots, N\} \quad 1 \\ 0 & \text{o.w.} \end{cases}$$

4) X: largest no. drawn on the tickets in n draws

$E_X = \{1, 2, \ldots, N\}$ 1

$P(X \le x) = \left(\frac{x}{N}\right)^n$ → 1

½

CD.f –

$$P(X \le x) = \begin{cases} 0 & x \le 1 \\ \left(\frac{\lfloor x \rfloor}{N}\right)^n & 1 \le x \le N \\ 1 & x > N \end{cases}$$ → ½ + ½

P.m.f –

$P(X = x) = F_X(x) - F_X(x^-)$ ½

$= P(X \le x) - P(X \le x-1)$

$= \left(\frac{x}{N}\right)^n - \left(\frac{x-1}{N}\right)^n = \frac{x^n - (x-1)^n}{N^n}$ 1

5)

$$f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

$Y = X^{1/\alpha}$ $\qquad E_Y = [0, \infty)$

$F_Y(y) = P(Y \le y) = P(X^{1/\alpha} \le y) = P[X \le y^\alpha]$

$= F_X(y^\alpha)$ 1

$F_X(x) = 1 - e^{-2x}$

$F_X(y^\alpha) = 1 - e^{-2y^\alpha}$ 1

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-2y^\alpha} & y \ge 0 \end{cases}$$ 1

1

PDF – $f_Y(y) = \frac{d}{dy}\left(F_Y(y)\right) = \begin{cases} 2\alpha y^{\alpha-1} e^{-2y^\alpha} & y > 0 \\ 0 & \text{o.w.} \end{cases}$

(6). Chebyshev Inequality is used here.

Suppose that R.V. has finite first two moments. If $\mu = E(X)$ and $\sigma^2 = Var(X)$, then for any $a > 0$

$$P(\{|X - \mu| \geq a\}) \leq \frac{\sigma^2}{a^2}.$$

Let X be the no. of times '6' appears on the die, then

$$X \sim Bin\left(m, \frac{1}{6}\right) \quad 1$$

$$P(|X - \mu| < a) \geq 1 - \frac{\sigma^2}{a^2} \quad 1$$

$$\mu = m \cdot \frac{1}{6} = \frac{m}{6}$$

$$\sigma^2 = m \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5m}{36}$$

Applying above inequality

$$P\left(\left|X - \frac{m}{6}\right| < \sqrt{m}\right) \geq 1 - \frac{\sigma^2}{m} = 1 - \frac{5m/36}{m} = \frac{31}{36}$$

$$P\left(\frac{m}{6} - \sqrt{m} < X < \frac{m}{6} + \sqrt{m}\right) \geq \frac{31}{36} \quad 1$$