

Indian Institute of Information Technology Allahabad
Mid Sem Examination

Course Name: Probability and Statistics
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Important Instructions:

1. This question paper contains two pages.
 2. Attempt all questions. There is no credit for a solution if the appropriate work is not shown, even if the answer is correct. All the notations are standard and same as used in the lecture notes.
 3. Attempt all the parts of a question at the same place. Parts done separately will not be graded.
 4. The use of any electronic gadgets, including smart watches, calculators is not allowed.
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1. A drawer contains two coins. One is an unbiased coin, which when tossed, is equally likely to turn up heads or tails. The other is a biased coin, which will turn up heads with probability p and tails with probability $1 - p$. One coin is selected at random from the drawer. The selected coin is tossed n times. Given that the coin turns up heads k times and tails $n - k$ times, what is the probability that the coin is biased? [5]
Let E_1 be the event that the coin is fair and E_2 be the event that the coin is biased.

Solution: Let E be the event that out of n tosses, head occurs k times and tail occurs $n - k$ times.

We have, $P(E_1) = P(E_2) = \frac{1}{2}$ [1]

For a fair coin, $P(E | E_1) = \binom{n}{k} \left(\frac{1}{2}\right)^n$ [1]

For a biased coin, $P(E | E_2) = \binom{n}{k} p^k (1 - p)^{n-k}$ [1]

We need to find $P(E_2 | E)$.

By the definition of conditional probability,

$$\begin{aligned}
P(E_2 \mid E) &= \frac{P(E_2 \cap E)}{P(E)} \\
&= \frac{P(E_2)P(E \mid E_2)}{P(E_1)P(E \mid E_1) + P(E_2)P(E \mid E_2)} \\
&= \frac{\frac{1}{2} \binom{n}{k} p^k (1-p)^{n-k}}{\frac{1}{2} \binom{n}{k} \left(\frac{1}{2}\right)^n + \frac{1}{2} \binom{n}{k} p^k (1-p)^{n-k}} \\
&= \frac{p^k (1-p)^{n-k}}{\left(\frac{1}{2}\right)^n + p^k (1-p)^{n-k}}.
\end{aligned}$$

Hence, $P(E_2 \mid E) = \frac{p^k (1-p)^{n-k}}{\left(\frac{1}{2}\right)^n + p^k (1-p)^{n-k}}$ [2]

2. Let X be a continuous random variable following exponential distribution with parameter $\lambda > 0$. Define $Y = [X]$, where $[x]$ denote the greatest integer less than or equal to x . Then, [5]

(a) find the probability mass function (p.m.f.) of Y .

Solution: Let $X \sim \exp(\lambda)$ and $Y = [X]$, so $Y = h(X)$ with $h(x) = [x]$. $f_X(x) = \lambda e^{-\lambda x}$, $x > 0$ [1]

The support of Y is $E_Y = \{[x] : x \in E_X\} = \mathbb{N} \cup \{0\}$.

For $y \in \mathbb{N} \cup \{0\}$:

$$\begin{aligned}
f_Y(y) &= P(Y = y) \\
&= P([X] = y) \\
&= P(h(X) = y) = P(X \in h^{-1}(y)) \\
&= P(X \in [y, [y] + 1))
\end{aligned}$$
 [1]

Let $\lfloor y \rfloor = i$,

$$\begin{aligned}
 P(X \in [i, i+1)) &= \lambda \int_i^{i+1} e^{-\lambda x} dx \\
 &= \frac{\lambda}{(-\lambda)} \left[e^{-\lambda x} \right]_i^{i+1} \\
 &= e^{-\lambda i} - e^{-\lambda(i+1)} \\
 &= e^{-\lambda i} (1 - e^{-\lambda}). \tag{1}
 \end{aligned}$$

$$f_Y(y) = \begin{cases} e^{-\lambda y} (1 - e^{-\lambda}), & y \in \{0, 1, 2, \dots\}, \\ 0, & \text{otherwise.} \end{cases} \tag{1/2 + 1/2}$$

(Using CDF method)- Let $X \sim \exp(\lambda)$ and $Y = \lfloor X \rfloor$.

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0 \tag{1}$$

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}$$

The CDF of Y is

$$\begin{aligned}
 F_Y(y) &= P(Y \leq y) = P(\lfloor X \rfloor \leq y) = P(X < \lfloor y \rfloor + 1) \tag{1} \\
 &= F_X(\lfloor y \rfloor + 1) \\
 &= 1 - e^{-\lambda(\lfloor y \rfloor + 1)}
 \end{aligned}$$

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-\lambda(\lfloor y \rfloor + 1)}, & y \geq 0 \end{cases} \tag{1}$$

Therefore,

$$\begin{aligned} P(Y = y) &= F_Y(y) - F_Y(y - 1) \\ &= \left(1 - e^{-\lambda(\lfloor y \rfloor + 1)}\right) - \left(1 - e^{-\lambda \lfloor y \rfloor}\right) \\ &= e^{-\lambda y} - e^{-\lambda(y+1)} \\ &= e^{-\lambda y} (1 - e^{-\lambda}) \end{aligned}$$

$$\Rightarrow f_Y(y) = \begin{cases} e^{-\lambda y} (1 - e^{-\lambda}), & y \in \{0, 1, 2, \dots\} \\ 0, & \text{otherwise} \end{cases} \quad \left(\frac{1}{2} + \frac{1}{2}\right)$$

(b) Identify Y as a known discrete distribution and state its parameters.

Solution: Let $p = 1 - e^{-\lambda}$ and $q = e^{-\lambda}$. Then

$$f_Y(i) = p q^i, \quad i = 0, 1, 2, \dots \quad \text{Thus } Y \sim \text{Geo}(p) \quad [1]$$

3. Let X be a continuous random variable with probability density function (p.d.f.) given by: [6]

$$f_X(x) = \frac{e^{-|x|}}{2}, \quad -\infty < x < \infty.$$

(a) Find the moment generating function $M_X(t)$ and write the domain where $M_X(t)$ exists.

(b) Calculate the mean $E(X)$ and the variance $\text{Var}(X)$.

Solution: Given the probability density function

$$f_X(x) = \frac{e^{-|x|}}{2}, \quad -\infty < x < \infty.$$

The moment generating function of X is

$$M_X(t) = E(e^{tX}).$$

Therefore,

$$\begin{aligned}M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{e^{-|x|}}{2} dx \\&= \frac{1}{2} \int_{-\infty}^0 e^{tx} e^x dx + \frac{1}{2} \int_0^{\infty} e^{tx} e^{-x} dx \\&= \frac{1}{2} \int_{-\infty}^0 e^{(t+1)x} dx + \frac{1}{2} \int_0^{\infty} e^{(t-1)x} dx\end{aligned}\tag{1}$$

For the integrals to converge, we require

$$t + 1 > 0$$

and

$$t - 1 < 0.$$

Hence,

$$t > -1 \quad \text{and} \quad t < 1.$$

Thus, $\boxed{t \in (-1, 1)}$.[1]

Now,

$$\begin{aligned}M_X(t) &= \frac{1}{2} \left[\frac{e^{(t+1)x}}{t+1} \right]_{-\infty}^0 + \frac{1}{2} \left[\frac{e^{(t-1)x}}{t-1} \right]_0^{\infty} \\&= \frac{1}{2} \left(\frac{1}{t+1} - \frac{1}{t-1} \right) \\&= \frac{1}{2} \left(\frac{(t-1) - (t+1)}{(t+1)(t-1)} \right) \\&= \frac{1}{1-t^2}.\end{aligned}$$

Therefore, $\boxed{M_X(t) = \frac{1}{1-t^2}, \quad -1 < t < 1.}$ [1]

Mean: We know that

$$E(X) = M'_X(0).$$

Differentiating,

$$\begin{aligned} M'_X(t) &= \frac{d}{dt} \left(\frac{1}{1-t^2} \right) \\ &= \frac{2t}{(1-t^2)^2}. \end{aligned}$$

Hence,

$$E(X) = M'_X(0) = 0.$$

Therefore, $\boxed{E(X) = 0.}$ [1]

Variance: We have

$$E(X^2) = M''_X(0).$$

Differentiating again,

$$\begin{aligned} M''_X(t) &= \frac{d}{dt} \left(\frac{2t}{(1-t^2)^2} \right) \\ &= \frac{2}{(1-t^2)^2} + \frac{8t^2}{(1-t^2)^3} \\ &= \frac{2(1+3t^2)}{(1-t^2)^3}. \end{aligned}$$

Therefore,

$$E(X^2) = M''_X(0) = 2. \quad [1]$$

Now,

$$\text{Var}(X) = E(X^2) - [E(X)]^2.$$

Hence,

$$\text{Var}(X) = 2 - 0^2 = 2.$$

Therefore, $\boxed{\text{Var}(X) = 2}$. [1]

4. Does there exist a random variable X with mean 5 and variance 9 such that $P(-1 < X < 11) = 0.6$? Justify your answer. [4]

Solution: Given,

$$P(-1 < X < 11) = 0.6$$

Now, $P(-6 < X - 5 < 6) = P(|X - 5| < 6) = 1 - P(|X - 5| \geq 6)$ [1]

Applying Chebyshev's Inequality, $P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$ [1]

we have

$$P(|X - 5| \geq 6) \leq \frac{1}{4}$$

$$1 - P(|X - 5| \geq 6) \geq 1 - \frac{1}{4} = 0.75$$

$$P(|X - 5| < 6) \geq 0.75$$
 [1]

which is a contradiction as problem states $P(-1 < X < 11) = 0.6$.

So, no such random variable can exist. [1]

5. Let $X \sim N(\mu, \sigma^2)$ so that $P(X < 89) = 0.881$ and $P(X < 94) = 0.985$. Find μ and σ^2 . [5]

Solution:

$$X \sim N(\mu, \sigma^2), \quad P(X < 89) = 0.881, \quad P(X < 94) = 0.985.$$

$$P\left(\frac{X - \mu}{\sigma} < \frac{89 - \mu}{\sigma}\right) = 0.881 \Rightarrow \Phi\left(\frac{89 - \mu}{\sigma}\right) = 0.881$$
 [1]

$$\Phi\left(\frac{89 - \mu}{\sigma}\right) = 0.881 \quad \Rightarrow \quad \frac{89 - \mu}{\sigma} = 1.18 \quad [1]$$

$$\Phi\left(\frac{94 - \mu}{\sigma}\right) = 0.985 \quad \Rightarrow \quad \frac{94 - \mu}{\sigma} = 2.17 \quad [1]$$

$$89 - \mu = 1.18\sigma, \quad 94 - \mu = 2.17\sigma$$

$$5 = (2.17 - 1.18)\sigma \quad \Rightarrow \quad \sigma = \frac{5}{0.99} \approx 5.05$$

$$\mu = 89 - 1.18(5.05) \approx 83.04 \quad [1]$$

$$\sigma^2 = (5.05)^2 \approx 25.51 \quad [1]$$

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361