Bernoulli, Binomial and Uniform Distributions

Let $(\mathcal{S}, \Sigma, P)$ be a probability space corresponding to a random experiment $\mathcal{E}$.

- Each repetition of the random experiment $\mathcal{E}$ will be called a trial.
- We say that a collection of trials forms a collection of independent trials if any collection of corresponding events forms a collection of independent events.

1. Bernoulli Distribution

A random experiment is said to be a Bernoulli experiment if its each trial results in just two possible outcomes, labeled as success (s) and failure (f). Each repetition of a Bernoulli experiment is called a Bernoulli trial. For example, consider a sequence of random rolls of a fair dice. In each roll of the dice a person bets on occurrence of upper face with six dots. Let the event of occurrence of upper face with six dots be denoted by E. Here, in each trial, one is only interested in the occurrence or non-occurrence of the event E. In such situations, the occurrence of event E will be label as a success and the non-occurrence of event E will be label as a failure.

For a Bernoulli trial, the sample space is $\mathcal{S} = \{s, f\}$, the event space is $\Sigma = \mathcal{P}(\mathcal{S})$ and the probability function is $P : \Sigma \longrightarrow \mathbb{R}$ defined by $P(\{s\}) = p$, $P(\{f\}) = 1 - p$, $P(\{\emptyset\}) = 0$ and $P(\{\mathcal{S}\}) = 1$, where $p \in (0, 1)$ is a fixed real number and it is the probability of success of the trial. Define the random variable $X : \mathcal{S} \longrightarrow \mathbb{R}$ by

$$X(w) = \begin{cases} 1, & \text{if } w = s \\ 0, & \text{if } w = f \end{cases}$$

Then the r.v. X is of discrete type with the support $E_X = \{0, 1\}$ and the p.m.f.

$$f_X(x) = P(\{X = x\}) = \begin{cases} 1 - p, & \text{if } x = 0 \\ p, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases} . \tag{1}$$

The random variable X is called a Bernoulli random variable and the distribution with p.m.f. (1) is called a Bernoulli distribution with success probability $p \in (0, 1)$.

The d.f. of X is given by

$$F_X(x) = P(\{X \leq x\}) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - p, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x \geq 1 \end{cases}$$

Now, the expectation of X is $E(X) = \sum\limits_{x \in \{0,1\}} x f_X(x) = p$ and $E(X^2) = \sum\limits_{x \in \{0,1\}} x^2 f_X(x) = p$. Thus the variance is $Var(X) = p - p^2 = p(1-p)$. Also the moment generating functions is

$$M_X(t) = E(e^{tX}) = \sum_{x \in \{0,1\}} e^{tx} f_X(x) = p(e^t - 1) + 1, \ \forall \ t \in \mathbb{R}$$

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2. Binomial Distribution

Consider a sequence of n independent Bernoulli trials with probability of success (s) in each trial being $p \in (0,1)$. In this case, the sample space is $\mathcal{S} = \{(w_1, w_2, \ldots, w_n) \mid w_i \in \{s, f\}, i = 1, 2, \ldots, n\}$, where w_i represents the outcome of the i-th Bernoulli trial and the event space is $\Sigma = \mathcal{P}(\mathcal{S})$. Define the random variable $X : \mathcal{S} \longrightarrow \mathbb{R}$ by

$$X((w_1, w_2, \ldots, w_n)) = \text{number of successes among } w_1, w_2, \ldots, w_n$$

Clearly, Im X $= \{0, 1, 2, \cdots, n\}$ and $P(\{X = x\}) = 0$, if $x \notin \{0, 1, 2, \cdots, n\}$. For $x \in \{0, 1, 2, \cdots, n\}$

$$\begin{aligned} P(\{X = x\}) &= P(\{(w_1, w_2, \ldots, w_n) \in S \mid X(w_1, w_2, \ldots, w_n) = x\}) \\ &= \sum_{(w_1, w_2, \ldots, w_n) \in S_x} P((w_1, w_2, \ldots, w_n)), \end{aligned}$$

where $S_x = \{(w_1, w_2, \ldots, w_n) \mid x \text{ of } w_i's \text{ are } s \text{ and remaining } n - x \text{ of } w_i's \text{ are } f\}$.

For $x \in \{0, 1, 2, \cdots, n\}$ and $(w_1, w_2, \ldots, w_n) \in S_x$,

$$P((w_1, w_2, \ldots, w_n)) = p^x (1-p)^{n-x},$$

since trials are independent and $P(\{s\}) = p$ & $P(\{f\}) = 1 - p$. Therefore, $x \in \{0, 1, 2, \cdots, n\}$,

$$P(\{X = x\}) = \sum_{(w_1, w_2, \ldots, w_n) \in S_x} p^x (1-p)^{n-x} = \binom{n}{x} p^x (1-p)^{n-x}.$$

Thus the r.v. X is of discrete type with support $E_X = \{0, 1, 2, \cdots, n\}$ and p.m.f.

$$f_X(x) = P(\{X = x\}) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & \text{if } x \in \{0, 1, 2, \cdots, n\} \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

The random variable X is called a Binomial random variable with n trials and success probability $p \in (0,1)$ and it is written as $X \sim \text{Bin}(n, p)$. The probability distribution with the p.m.f. (2) is called a Binomial distribution with n trials and success probability $p \in (0,1)$. It is clear that $\sum\limits_{x \in E_X} f_X(x) = \sum\limits_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = (p + (1-p))^n = 1$

Now, the expectation of $X \sim \text{Bin}(n, p)$ is

$$\begin{aligned}
E(X) &= \sum_{x \in E_X} x f_X(x) \\
&= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \frac{xn!}{(n-x)x!} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} \frac{n!}{(n-x)(x-1)!} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)(x-1)!} p^{(x-1)} (1-p)^{n-x} \\
&= np \sum_{x=0}^{n-1} \binom{n-1}{x} p^x (1-p)^{n-1-x} \\
&= np(p + (1-p))^{(n-1)} = np
\end{aligned}$$

Now, the moment generating function of $X \sim \text{Bin}(n, p)$ is

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \sum_{x \in E_X} e^{tx} f_X(x) \\
&= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\
&= (pe^t + (1-p))^n, \ t \in \mathbb{R}
\end{aligned}$$

Therefore,

$M_X^{(1)}(t) = npe^t(pe^t + (1-p))^{(n-1)}, \ t \in \mathbb{R};$
$M_X^{(2)}(t) = npe^t(pe^t + (1-p))^{(n-1)} + n(n-1)p^2e^{2t}(pe^t + (1-p))^{(n-2)}, \ t \in \mathbb{R};$
$E(X) = M_X^{(1)}(0) = np;$
$E(X^2) = M_X^{(2)}(0) = np + n(n-1)p^2;$
and $Var(X) = E(X^2) - (E(X))^2 = np(1-p).$

Example 1. *Four fair coins are flipped. If the outcomes are assumed independent, what is the probability that two heads and two tails are obtained?*

Solution: Let us label the occurrence of a head in a trial as success and label the occurrence of a tail in a trial as failure. Let X be the number of successes (i.e. heads) that appear. Then $X \sim \text{Bin}(4, \frac{1}{2})$. Hence the required probability is $P(X = 2) = \binom{4}{2}(\frac{1}{2})^2(\frac{1}{2})^2 = \frac{3}{8}$.

Example 2. *A fair dice is rolled six times independently. Find the probability that on two occasions we get an upper face with 2 or 3 dots.*

Solution: Let us label the occurrence of an upper face having 2 or 3 dots as success and label the occurrence of any other face as failure. Let X be the number of occasions on which we get success (i.e., an upper face having 2 or 3 dots). Then $X \sim \text{Bin}(6, \frac{1}{3})$. Hence the required probability is $P(X = 2) = \binom{6}{2}(\frac{1}{3})^2(\frac{2}{3})^4 = \frac{80}{243}$.

3. Discrete Uniform Distribution

For a given positive integer $N(\geq 2)$ and real numbers $x_1 < x_2 < \cdots < x_N$, a random variable X of discrete type is said to follow a discrete uniform distribution on the set $\{x_1, x_2, \ldots, x_N\}$ (written as $X \sim U(\{x_1, x_2, \ldots, x_N\})$) if the support of X is $E_X = \{x_1, x_2, \ldots, x_N\}$ and its p.m.f. is given by

$$f_X(x) = P(\{X = x\}) = \begin{cases} \frac{1}{N}, & \text{if } x \in E_X = \{x_1, x_2, \ldots, x_N\} \\ 0, & \text{otherwise} \end{cases}$$

.

Now, for $r \in \{1, 2, \cdots\}$, $E(X^r) = \frac{1}{N}\sum_{i=1}^{N} x_i^r$. Therefore, the mean $E(X) = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $Var(X) = \frac{1}{N}\sum_{i=1}^{N}(x_i - E(X))^2$. Also the m.g.f. is $M_X(t) = E(e^{tX}) = \frac{1}{N}\sum_{i=1}^{N} e^{tx_i}$, $t \in \mathbb{R}$.

Now, suppose that $X \sim U(\{1, 2, \ldots, N\})$. Then

$$E(X) = \frac{1}{N}\sum_{i=1}^{N} i = \frac{N+1}{2},$$

$$E(X^2) = \frac{1}{N}\sum_{i=1}^{N} i^2 = \frac{(N+1)(2N+1)}{6},$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{N^2-1}{12}.$$

Also the m.g.f. of $X \sim U(\{1, 2, \ldots, N\})$ is

$$M_X(t) = E(e^{tX}) = \frac{1}{N}\sum_{i=1}^{N} e^{it} = \begin{cases} \frac{e^t(e^{Nt}-1)}{e^t-1}, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0 \end{cases}$$