Testing Statistical Hypotheses

Hypothesis: A statistical hypothesis is an assertion about the probability distribution of population.

Null Hypothesis: It is a first tentative specification about the probability model. It is denoted by H_0. For example, $H_0 : p = 0.75$, $H_0 : \mu_1 = \mu_2$.

Alternative Hypothesis: Another possibility in contrast to the null hypothesis is called alternative hypothesis. It is denoted by H_1 or H_a. For example, $H_1 : p > 0.75$, $H_1 : \mu_1 \neq \mu_2$.

Problem of Testing of Hypothesis: Let Θ be the parameter space, and Θ_0 and Θ_1 be its disjoint subsets (may not be complementary always).
Let the null and alternative hypotheses be defined as $H_0 : \theta \in \Theta_0$ and $H_1 : \theta \in \Theta_1$. Now, we will define a test of statistical hypotheses.

Test of Statistical Hypotheses: A test of statistical hypothesis is a procedure to decide whether to accept or reject the null hypothesis.
We take observations from the given population and based on this, we take the decision to accept or reject H_0.

For example, consider a die and let p be the probability of occurrence of a six. We want to test whether the die is fair, i.e., we want to test $H_0 : p = \frac{1}{6}$ against $H_1 : p \neq \frac{1}{6}$. To test this, let us toss the die, say $n = 60$ times, and let X be the number of sixes out of 60 tosses. Based on the sample results, one may define a test such as, accept H_0, if $X = 9, 10, 11$, reject otherwise.

Simple Hypothesis & Composite Hypothesis: A hypothesis is called simple if it completely specifies a probability model otherwise it is called a composite hypothesis. For example $H_0 : \mu = \mu_0$ is simple, whereas, $H_0^* : \mu > \mu_0$ is composite.

Type I Error: Rejecting H_0, when it is true is known as type I error. The probability of type I error is known as α.

Type II Error: Accepting H_0, when it is false is known as type II error. The probability of type II error is known as β.

Power of Test: $1 - \beta$ is known as power of the test.

The consequences of both the errors are different. In an ideal test procedure both α and β should be minimum. However, simultaneous minimization of both α and β is not possible. Therefore, we try to fix an upper bound on one error and then find a test procedure for which the second probability is minimum. A standard convention is to fix α and minimize β for that fixed α.

Example: Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, 1)$. We want to test $H_0 : \mu = -1/2$ against $H_1 : \mu = 1/2$.
Here, the acceptance region is $A = (-\infty, 0]$, i.e., accept H_0 if $\bar{X} \leq 0$. The rejection region is $R = (0, \infty)$, i.e., reject H_0 if $\bar{X} > 0$. Now, we calculate both the errors.

$$\begin{aligned} \alpha = & \text{ Prob(Type I error)= Prob(Rejecting } H_0\text{, when it is true)} \\ = & \; P_{\mu=\frac{-1}{2}}(\bar{X} > 0) = P_{\mu=\frac{-1}{2}}(\sqrt{n}(\bar{X} + \tfrac{1}{2}) > \tfrac{\sqrt{n}}{2}) = P(Z > \tfrac{\sqrt{n}}{2}) \\ = & \; P(Z > 2) = 0.0228, \qquad \text{for } n = 16. \end{aligned}$$

Here, α and β are same.

Now let us modify the test procedure. Let the acceptance and rejection region be $A^* = \{\bar{X} < \frac{-1}{4}\}$ and $R^* = \{\bar{X} \geq \frac{-1}{4}\}$, respectively. Therefore, the probability of type I and type II errors are

$$\alpha^* = P_{\mu=\frac{-1}{2}}(\bar{X} \geq \frac{-1}{4}) = P_{\mu=\frac{-1}{2}}(\sqrt{n}(\bar{X}+\frac{1}{2}) > \frac{\sqrt{n}}{4}) = P(Z \geq \frac{\sqrt{n}}{4}) = 0.1587,$$

for $n = 16$.

$$\beta^* = P_{\mu=\frac{1}{2}}(\bar{X} < \frac{-1}{4}) = P(Z < -3) = 0.0013,$$

for $n = 16$.
Here, we observe that $\beta^* < \beta$ but $\alpha^* > \alpha$. Hence, it is clear that the simultaneous minimization of both the errors α and β is not possible.

Interval Estimation

In the previous lectures, we have discussed the point estimation where we have a single value as an estimator for the unknown value of parameter. Now, we will discuss interval estimation where we have an interval as an estimator for the unknown parameter.
Let $\underline{X} = (X_1, \cdots, X_n)$ be a random sample from a population with distribution $P_{\underline{\theta}}, \underline{\theta} \in \Theta \subset \mathbb{R}^k$. A family of subsets $S(\underline{X})$ of Θ is said to be a family of confidence sets at confidence level $(1-\alpha)$ if

$$P(\underline{\theta} \in S(\underline{X})) \geq 1 - \alpha, \ \forall \ \theta \in \Theta.$$

In case of $k = 1$, $S(\underline{X}) = (a(\underline{X}), b(\underline{X}))$ is said to be $(1-\alpha)$ level confidence interval for θ if

$$P(a(\underline{X}) < \theta < b(\underline{X})) \geq 1 - \alpha,\ \forall\, \theta \in \Theta.$$

Example 1: Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, 1)$. Let us construct a confidence interval for μ. We know, $\bar{X}$ follows $N(\mu, \frac{1}{n})$. We consider the confidence interval of the type $(\bar{X} - c_1, \bar{X} + c_2)$, so that

$$P(\bar{X} - c_1 < \mu < \bar{X} + c_2) \geq 1 - \alpha$$

$$P(-c_2 < \bar{X} - \mu < c_1) \geq 1 - \alpha$$

$$P(-\sqrt{n}c_2) < \sqrt{n}(\bar{X} - \mu) < \sqrt{n}c_1 \geq 1 - \alpha$$

$$P(-\sqrt{n}c_2 < Z < \sqrt{n}c_1) \geq 1 - \alpha$$

If we choose $c_1 = c_2 = c$, then $c\sqrt{n} = z_{\alpha/2}$.
Therefore $c_1 = c_2 = \frac{1}{\sqrt{n}} z_{\alpha/2}$.
Hence, $\left(\bar{X} - \frac{1}{\sqrt{n}} z_{\alpha/2}, \bar{X} + \frac{1}{\sqrt{n}} z_{\alpha/2}\right)$ is $(1-\alpha)$ level confidence interval for μ.