Moments, Covariance and Correlation Coefficient

Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a $n-$dimensional ($n \geq 2$) random vector and $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function such that $\psi^{-1}(A) \in \mathbb{B}_{\mathbb{R}^n}$, for all $A \in \mathbb{B}_{\mathbb{R}}$. Suppose $E(\psi(\underline{X}))$ is finite.

(1) If $\underline{X}$ is of discrete type with joint p.m.f. $f_{\underline{X}}$ and support $E_{\underline{X}}$, then
$$E(\psi(\underline{X})) = \sum_{(x_1,x_2,\ldots,x_n)\in E_{\underline{X}}} \psi(x_1, x_2, \ldots, x_n)) f_{\underline{X}}(x_1, x_2, \ldots, x_n).$$

(2) If $\underline{X}$ is of continuous type with joint p.d.f. $f_{\underline{X}}$, then
$$E(\psi(\underline{X})) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \psi(x_1, x_2, \ldots, x_n) f_{\underline{X}}(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots\, dx_n.$$

(3) For nonnegative integers $k_1, k_2, \ldots, k_n$, let $\psi(x_1, x_2, \ldots, x_n) = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$. Then
$$\mu'_{k_1,k_2,\ldots,k_n} = E(\psi(\underline{X})) = E(X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}),$$
provided it is finite, is called the joint moment of order $k_1 + k_2 + \cdots + k_n$ of $\underline{X} = (X_1, X_2, \ldots, X_n)$.

(4) For $n = 2$, let $\psi(x_1, x_2) = (x_1 - E(X_1))(x_2 - E(X_2))$. Then
$$Cov(X_1, X_2) = E\Big((X_1 - E(X_1))(X_2 - E(X_2))\Big),$$
provided it is finite, is called the covariance between X_1 and X_2.

Note: By the definition of covariance, it is easy to see
$$Cov(X_1, X_1) = Var(X_1);$$
$$Cov(X_1, X_2) = Cov(X_2, X_1);$$
$$Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2).$$

Theorem 1. *Let $\underline{X} = (X_1, X_2)$ and $\underline{Y} = (Y_1, Y_2)$ be two random vectors and a_1, a_2, b_1, b_2 be real constants. Then, provided the involved expectations are finite,*

(1) $E(a_1X_1 + a_2X_2) = a_1E(X_1) + a_2E(X_2)$;

(2) $Cov(a_1X_1 + a_2X_2, b_1Y_1 + b_2Y_2) = a_1b_1\, Cov(X_1, Y_1) + a_1b_2\, Cov(X_1, Y_2) + a_2b_1\, Cov(X_2, Y_1) + a_2b_2\, Cov(X_2, Y_2) = \sum_{i=1}^{2} \sum_{j=1}^{2} a_ib_jCov(X_i, Y_j)$.
In particular,
$Var(a_1X_1 + a_2X_2) = Cov(a_1X_1 + a_2X_2, a_1X_1 + a_2X_2) = a_1^2Var(X_1) + a_2^2Var(X_2) + 2a_1a_2Cov(X_1, X_2)$.

Definition 2. (1) *The correlation coefficient between random variables X and Y is defined by*
$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}},$$
provided $0 < Var(X), Var(Y) < \infty$.

(2) *The random variables X and Y are said to be uncorrelated if $Cov(X, Y) = 0$.*

Note: By definition, it is clear that if X and Y are independent random variables, then they are uncorrelated but converse need not be true.

Theorem 3. *Let X and Y be two random variables. Then, provided the involved expectations are finite,*

(1) $(E(XY))^2 \leq E(X^2)E(Y^2)$. *Moreover, $(E(XY))^2 = E(X^2)E(Y^2)$ if and only if $P(Y = cX) = 1$ or $P(X = cY) = 1$, for some $c \in \mathbb{R}$.*
This inequality is know as Cauchy-Schwarz inequality for random variables.

(2) $|\rho(X,Y)| \le 1$. *To prove it, apply (1) on random variables* $X' = X - E(X)$ *and* $Y' = Y - E(Y)$.

Example 4. *Let* $\underline{Z} = (X, Y)$ *be a random vector of discrete type with joint p.m.f.*

$$f(x,y) = \begin{cases} p_1, & \text{if } (x,y) = (-1,1) \\ p_2, & \text{if } (x,y) = (0,0) \\ p_1, & \text{if } (x,y) = (1,1) \\ 0, & \text{otherwise} \end{cases}$$

where $p_1, p_2 \in (0,1)$ *and* $2p_1 + p_2 = 1$.

Then the support of $\underline{Z}, X$ *and* Y *are*

$$E_{\underline{Z}} = \{(-1,1),(0,0),(1,1)\}$$
$$E_X = \{-1,0,1\}$$
and
$$E_Y = \{0,1\},$$

respectively. Clearly $E_{\underline{Z}} \neq E_X \times E_Y$. *So,* X *and* Y *are not independent.*

Now,

$$E(XY) = \sum_{(x,y)\in E_{\underline{Z}}} xyf(x,y) = 0;$$
$$E(X) = \sum_{(x,y)\in E_{\underline{Z}}} xf(x,y) = 0;$$
$$E(Y) = \sum_{(x,y)\in E_{\underline{Z}}} yf(x,y) = 2p_1;$$
$$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = 0 \Rightarrow \rho(X,Y) = 0$$

This shows that X *and* Y *are uncorrelated but not independent.*

We can also show that X *and* Y *are not independent by another way.*

The marginal p.m.f. of X *is*

$$f_X(x) = \begin{cases} \sum\limits_{y\in R_x} f(x,y), & \text{if } x \in \{-1,0,1\} \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} p_1, & \text{if } x = -1 \\ p_2, & \text{if } x = 0 \\ p_1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, the marginal p.m.f. of Y *is*

$$f_Y(y) = \begin{cases} \sum\limits_{x\in R_y} f(x,y), & \text{if } y \in \{0,1\} \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} p_2, & \text{if } x = 0 \\ 2p_1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Since $f(-1,1) \neq f_X(-1)f_Y(1)$, X *and* Y *are not independent.*

Example 5. *Let* $\underline{Z} = (X, Y)$ *be a random vector of continuous type with joint p.d.f.*

$$f(x,y) = \begin{cases} 1, & \text{if } 0 < |y| \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Now,

$$
\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy = \int_{0}^{1}\int_{-x}^{x} xy\,dy\,dx = 0;\\
E(X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dx\,dy = \int_{0}^{1}\int_{-x}^{x} x\,dy\,dx = \frac{2}{3};\\
E(Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dx\,dy = \int_{0}^{1}\int_{-x}^{x} y\,dy\,dx = 0;\\
&\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = 0 \Rightarrow \rho(X,Y) = 0
\end{aligned}
$$

Thus X and Y are uncorrelated.

The marginal p.d.f. of X is

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy\\
&= \begin{cases} \int_{-x}^{x} dy, & \text{if } 0 < x < 1\\ 0, & \text{otherwise}\end{cases}\\
&= \begin{cases} 2x, & \text{if } 0 < x < 1\\ 0, & \text{otherwise}\end{cases}
\end{aligned}
$$

Similarly, the marginal p.d.f. of Y is

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx\\
&= \begin{cases} \int_{|y|}^{1} dx, & \text{if } -1 < y < 1\\ 0, & \text{otherwise}\end{cases}\\
&= \begin{cases} 1-|y|, & \text{if } -1 < y < 1\\ 0, & \text{otherwise}\end{cases}
\end{aligned}
$$

Since $f(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

We can also show that X and Y are not independent by another way. Then the support of $\underline{Z}, X$ and Y are

$$
\begin{aligned}
E_{\underline{Z}} &= \{(x,y) \in \mathbb{R}^2 \mid 0 < |y| \le x < 1\}\\
E_X &= (0,1)
\end{aligned}
$$

and

$$
E_Y = (-1,1),
$$

respectively. Clearly $E_{\underline{Z}} \neq E_X \times E_Y$. So, X and Y are not independent.

This example also shows that X and Y are uncorrelated but not independent.